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# Heterotic flux compactifications and their moduli

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## Abstract

We study supersymmetric compactification to four dimensions with non-zero H-flux in heterotic string theory. The background metric is generically conformally balanced and can be conformally Kähler if the primitive part of the H-flux vanishes. Analyzing the linearized variational equations, we write down necessary conditions for the existence of moduli associated with the metric. In a heterotic model that is dual to a IIB compactification on an orientifold, we find the metric moduli in a fixed H-flux background via duality and check that they satisfy the required conditions. We also discuss expressing the conditions for moduli in a fixed flux background using twisted differential operators.
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## 1. Introduction

One of the main appealing features of flux compactifications in string theory is its ability to stabilize moduli fields. From the perspective of the low energy effective action, turning on fluxes generates a classical potential [1–6] which typically lifts many of the moduli. Any remaining moduli may be further stabilized by non-perturbative effects [7,8]. (See [9] for an explicit example.) Such an approach assumes that the back-reaction of the non-zero fluxes on the background geometry is mild and that the geometry in the presence of fluxes can be continuously deformed from one without flux. This is certainly the case in well-studied M-theory [10] and type IIB [11]

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flux models where the background metric is modified simply by an additional warp factor when the fluxes are turned on.

However, in supersymmetric heterotic compactifications with non-zero H-flux, the back-reaction on the geometry can be drastic. Typically, the non-zero flux background geometry is non-Kähler, and more significantly, they are also topologically different from the zero-flux Calabi–Yau manifold [5,12]. Such background geometries necessarily cannot be continuously deformed from a Calabi–Yau geometry [6,13]. The excitations of the low-energy effective action, in particular the moduli fields, are no longer justifiably those present prior to the flux being turned on. An important issue then is to identify the moduli fields directly from the new flux geometry.

In this paper, we begin the study of the moduli fields in supersymmetric heterotic compactifications on six-dimensional non-Kähler geometries. Our analysis will be confined within the framework of supergravity. The study of moduli is then simply a linearized variational problem. The background fields satisfies certain constraining differential equations (e.g., Killing spinor equations or equations of motion) and the moduli corresponds to infinitesimal variation of the fields such that the constraining equations remain satisfied. Unfortunately, the differential equations though linear in variations typically couple the various variations in a complicated manner. Thus, solving the variational problem in general can be rather challenging.

In our study, we are aided by the fact that the currently known explicit models of heterotic flux backgrounds are dual to orientifold flux models in type IIB string theory [13,14]. Disregarding non-perturbative effects, the IIB orientifold models contain unlifted Kähler moduli in the presence of non-zero fixed $G_3$ fluxes. These IIB moduli can be mapped to give moduli in the heterotic flux model. A main motivation of ours is to understand these moduli solely within the context of the heterotic theory. The moduli are those associated with the metric and with the H-flux held fixed in the variation. Hence, our analysis of the heterotic moduli will concern mainly with the metric moduli and will also emphasize those in a fixed H-flux background.

We begin in Section 2 by giving an overview of the possible six-dimensional geometries that preserve $N = 1$ supersymmetry in four dimensions. In general, with non-zero H-flux, the metric is required to be conformally balanced. However, if the primitive part of the H-flux vanishes, then the metric is conformally Kähler. We then proceed to perform a linearized variation on the supersymmetry constraints for the background fields. We focus on the moduli associated with the metric. A metric variation will generically also require a simultaneous variation of the dilaton field. We will write down necessary conditions which the moduli associated with the metric must satisfy. In Section 3, we study the moduli of the heterotic flux models which are dual to models of IIB orientifolds with fluxes. We first write down the unlifted Kähler moduli in the IIB theory and then map them via supergravity duality rules to the heterotic theory. We find that these moduli satisfy the required conditions. We conclude in Section 4 commenting on some open questions and discussing the two conditions that the metric moduli in a fixed flux background must satisfied. Interestingly, the conditions may be reexpressed in terms of a twisted differential operator similar to those introduced in [15,16].

## 2. Moduli of heterotic compactifications with H-flux

We first review the constraints on supersymmetric flux compactification on $M^{3,1} \times K$, a four-dimensional Minkowski spacetime times a six-dimensional internal spin manifold [17–19] (see also [20–22]). The modern classification of supersymmetric backgrounds is given in terms of intrinsic torsion classes [18,19]. However, for the heterotic theory, the intrinsic torsion is deter-

mined solely by the H-flux. Thus, instead of linking the background geometries with intrinsic torsion classes, we emphasize the required geometry for each type of H-flux present. We take note of an interesting subset of solutions with conformally Kähler metrics on Calabi–Yau manifolds. We then proceed to study the moduli in backgrounds with non-zero H-flux.

## 2.1. Constraints on heterotic solutions

The background fields of a supersymmetric compactification in heterotic theory must obey

$$\delta\psi_M = \nabla_M\epsilon + \frac{1}{8}H_{MNP}\Gamma^{NP}\epsilon = 0, \tag{2.1}$$

$$\delta\lambda = \Gamma^M\partial_M\phi\epsilon + \frac{1}{12}H_{MNP}\Gamma^{MNP}\epsilon = 0, \tag{2.2}$$

$$\delta\chi = \Gamma^{MN}F_{MN}\epsilon = 0, \tag{2.3}$$

for some positive chirality 10d Majorana–Weyl spinor $\epsilon$.[1] In addition, the $H$-field obeys the modified Bianchi identity

$$dH = \frac{\alpha'}{4}\left(\mathrm{tr}\,R^{(-)}\wedge R^{(-)} - \mathrm{tr}\,F\wedge F\right). \tag{2.4}$$

As for quantization of the H-flux, with $dH \neq 0$, $H$ is in general not an element of the integer de Rham cohomology class $H^3(K,\mathbb{Z})$. However, if there exist three-cycles in regions of the manifold where $dH = 0$, then there is a Dirac quantization

$$\frac{1}{4\pi^2\alpha'}\int_\Gamma H \in \mathbb{Z}, \tag{2.5}$$

where $\Gamma$ is any three-cycle in those regions. Outside these region, the quantization condition is modified by the inclusion of a "defect" term $\delta = (1/16\pi^2)\int_\Gamma[\Omega_3(A) - \Omega_3(\omega^{(-)})]$ where $\Omega_3$ is the Chern–Simons three-form [23]. In general, this defect term does depend on the particular three-cycle $\Gamma$.

We note that the four equations (2.1)–(2.4) together are sufficient conditions for a background field configuration to satisfy the equations of motion of the 10d supergravity theory [24]. We shall take the spinor ansatz to be

$$\epsilon = \xi\otimes\eta + \bar{\xi}\otimes\bar{\eta}, \tag{2.6}$$

where $\xi$ and $\eta$ are 4d and 6d Weyl spinors, respectively, and $\bar{\xi} = (B^{(4)}\xi)^*$ and $\bar{\eta} = (B^{(6)}\eta)^*$ are the complex conjugated spinors.[2] $\eta$ will be taken to have positive chirality and unit normalized, $\eta^\dagger\eta = 1$. As a supersymmetry transformation parameter, $\eta$ must be everywhere non-vanishing on the internal manifold $K$. The presence of a single such spinor implies that there exists on $K$ a connection with $SU(3)$ holonomy. For $H \neq 0$, the $SU(3)$ holonomy connection is no longer the Levi-Civita connection. It has non-zero torsion and is precisely the torsional connection, $\omega^{(+)}{}_M{}^A{}_B = \omega_M{}^A{}_B + \frac{1}{2}H_M{}^A{}_B$, as given in (2.1), for which $\eta$ is covariantly constant.[3] We also

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[1] We mostly follow the conventions of [19]. The gauge field strength $F_{MN}$ is however here taken to be anti-Hermitian.

[2] $B^{(4)}$ and $B^{(6)}$ are the complex conjugation matrices that complex conjugate gamma matrices in four and six dimensions, respectively.

[3] A manifold with an $SU(3)$ holonomy connection has an $SU(3)$ structure group. Also, in the mathematics literature, the torsional connection is called the Bismut connection.

point out that the curvature tensor in (2.4) is defined with respect to the "minus" connection, $\omega^{(-)} = \omega - \frac{1}{2}H$. The justification comes from the worldsheet non-linear sigma model where the $\omega^{(-)}$ connection is required to preserve both worldsheet conformal invariance and spacetime supersymmetry [22].

We will focus on the implications of the first two constraint equations, (2.1) and (2.2). Together, they imply that $K$ is a complex manifold with the complex structure

$$J_m{}^n = -i\eta^\dagger \gamma_m{}^n \eta, \tag{2.7}$$

which satisfies $J_m{}^k J_k{}^n = -\delta_m{}^n$. The metric is Hermitian with respect to the complex structure, i.e., $g_{mn} = J_m{}^r J_n{}^s g_{rs}$, and is used to define the associated real Hermitian two-form $J_{mn} = J_m{}^k g_{kn}$.[4] As for the three-form flux, $H$, the two constraints imply the following equations

$$H_{mnp} = 3J_{[m}{}^r \nabla_{|r|} J_{np]}, \tag{2.8}$$

$$H_{mnp} J^{np} = -2\nabla_p J_m{}^p = -4J_m{}^p \partial_p \phi. \tag{2.9}$$

We will work in complex coordinates, $z^a$ and $\bar{z}^{\bar{a}}$, such that the complex structure takes the standard form, $J_a{}^b = i\delta_a{}^b$, $J_{\bar{a}}{}^{\bar{b}} = -i\delta_{\bar{a}}{}^{\bar{b}}$, and with other components zero. The Hermitian form, $J_{mn}$, is now easily seen to be a $(1, 1)$-form

$$J_{a\bar{b}} = -J_{\bar{b}a} = ig_{a\bar{b}}, \tag{2.10}$$

and the constraint on $H$ (2.8) simplifies to

$$H = i(\bar{\partial} - \partial)J, \tag{2.11}$$

where $J$ denotes the Hermitian form and for example, $H_{ab\bar{c}} = -i(\partial_a J_{b\bar{c}} - \partial_b J_{a\bar{c}})$. From (2.11), we see that the real three-form H-flux must be a $(2, 1)$- and $(1, 2)$-form with respect to the complex structure. Furthermore, with $H^{(2,1)} = -i[dJ]^{(2,1)}$, it is clear that a non-zero H-flux forces the internal manifold $K$ to be non-Kähler. As for (2.9), it describes the primitivity of $H$. We see that $H$ is only primitive (i.e., $H_{mnp} J^{np} = 0$) if and only if the dilaton is a constant. But as noted in [19,27], the equations of motion of $N = 1$ 10d supergravity can only be satisfied for a constant dilaton if also $H = 0$. Thus, for a non-zero H-flux background, $\phi$ must be non-constant and $H$ is necessarily non-primitive.[5] It is useful to decompose $H$ as a sum of its primitive $H^{\rm P}$ and non-primitive $H^{\rm NP}$ parts as follows

$$H_{mnp} = H^{\rm P}_{mnp} + H^{\rm NP}_{mnp} = \left(H_{mnp} - \frac{3}{4} J_{[mn} H_{p]rs} J^{rs}\right) + \frac{3}{4} J_{[mn} H_{p]rs} J^{rs}, \tag{2.12}$$

where $H^{\rm P}_{mnp} J^{np} = 0$. Applying (2.9), we have that

$$H^{\rm NP} = i(\bar{\partial} - \partial)\phi \wedge J = -*(d\phi \wedge J), \tag{2.13}$$

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[4] To avoid any confusion, we refer to $J_{mn}$ as the Hermitian form, following Joyce [25]. As in [26], it is also called the Kähler form even when the Kähler condition, $dJ = 0$, is not satisfied.

[5] The requirement that $H$ is non-primitive is a supergravity condition, valid at lowest order in $\alpha'$. Including $\alpha'$ corrections may possibly loosen this requirement. An example of a constant dilaton background that satisfy the supersymmetry constraints (2.1)–(2.3) but not the equation of motion is the Iwasawa manifold solution given in [18]. As noted in [19], the solution does not satisfy the modified Bianchi identity (2.4).

where the Hodge star, $*$, is defined with respect to the volume form $\frac{1}{3!} J \wedge J \wedge J$. And with (2.11), we find that $dJ$ also has a non-zero non-primitive part given by

$$dJ^{\text{NP}} = d\phi \wedge J. \tag{2.14}$$

Finally, the two constraints also imply the existence of a holomorphic three-form. Consider the three-form fermion bilinear

$$\Omega_{mnp} = \bar{\eta}^\dagger \gamma_{mnp} \eta, \tag{2.15}$$

which is a $(3, 0)$-form with respect to the complex structure of (2.7). Utilizing the two constraints, the anti-holomorphic derivative acting on $\Omega$ gives

$$\partial_{\bar{a}}(\Omega_{abc}) = 2\partial_{\bar{a}}\phi \, \Omega_{abc}, \tag{2.16}$$

and therefore, we have that $w_{abc} = e^{-2\phi} \Omega_{abc}$ is a holomorphic three-form. The existence of such a three-form is due to the internal manifold $K$ having $SU(3)$ holonomy with respect to the torsional connection. The $SU(3)$ holonomy also implies that $K$ has zero first Chern class, i.e., $c_1(K) = 0$.

The above constraints on the fields, (2.11), (2.9), and (2.16), can be expressed simply in terms of exterior derivatives and the Hodge star operator as follows [19]

$$H = *e^{2\phi} d(e^{-2\phi} J), \tag{2.17}$$

$$d(e^{-2\phi} * J) = d\left(\frac{e^{-2\phi}}{2} J \wedge J\right) = 0, \tag{2.18}$$

$$d(e^{-2\phi} \Omega) = 0. \tag{2.19}$$

The equivalence of (2.17) with (2.11) can be verified using (2.13) and that in six dimensions, a primitive $(p, q)$-form satisfies the self-duality relation $*A^{(p,q)} = i(-1)^q A^{(p,q)}$ for $p + q = 3$.[6] Eq. (2.18) characterizes the required type of the compactification metric. It specifies that there must exist on any $N = 1$ background geometry a universal closed $(2, 2)$-form, $e^{-2\phi}(J \wedge J)$. In particular, by making the conformal rescaling, $J = e^\phi \tilde{J}$, we obtain

$$d(\tilde{J} \wedge \tilde{J}) = 0, \tag{2.20}$$

which is the defining condition for a balanced metric [28]. Thus, we arrive at the statement that a supersymmetric heterotic flux compactification requires a conformally balanced metric.

An interesting and obvious subset of the balanced metrics are those that satisfy additionally the Kähler condition, i.e., $d\tilde{J} = 0$. For backgrounds with conformally Kähler metrics, the H-flux has the property from (2.11) and (2.13) that

$$H = i(\bar{\partial} - \partial)(e^\phi \tilde{J}) = i(\bar{\partial} - \partial)\phi \wedge (e^\phi \tilde{J}) = H^{\text{NP}}, \tag{2.21}$$

which implies that the primitive part of the H-flux, $H^{\text{P}}$, is zero. Indeed, for backgrounds with $H^{\text{P}} = 0$, the compactification metric is conformally Kähler. With also the presence of the holomorphic 3-form and $SU(3)$ holonomy, the underlying manifold is actually a Calabi–Yau manifold though the Ricci-flat metric is not the physically relevant metric here.

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[6] In our convention, $(*A)_{mnp} = \frac{1}{3!} \epsilon_{mnp}{}^{qrs} A_{qrs}$ with the Levi-Civita tensor $\epsilon_{mnpqrs}$ taken as the volume form $\frac{1}{3!} J \wedge J \wedge J$. This differs by a minus sign with the convention used in [4].

In summary, we see that the $N = 1$ background manifold is required to be Hermitian with zero first Chern class. The metric types can be characterized by the H-flux as follows:

$$H = H^{\text{NP}} + H^{\text{P}} \quad \text{conformally balanced metric,}$$
$$H = H^{\text{NP}} \quad \text{conformally Kähler metric,}$$
$$H = 0 \quad \text{Calabi–Yau metric.} \tag{2.22}$$

### 2.2. Deformations of heterotic solutions

For a heterotic flux solution, it is phenomenologically important to know the moduli space of the solution with $N = 1$ supersymmetry. In terms of the four-dimensional low energy effective theory, each modulus corresponds to a massless scalar field. Given a solution, the moduli space can be found by deforming the solution such that it continues to satisfy the $N = 1$ supersymmetry conditions discussed above.

There is at least one modulus in all heterotic supergravity compactifications at the classical level. This is the dilaton modulus [17] arising from shifting the dilaton by a constant, $\phi \to \phi + c$, and keeping all other fields fixed. Indeed, the supersymmetry conditions and also the equations of motion are invariant under a constant dilaton shift.

Besides the dilaton modulus, a generic compactification will have other moduli from varying the other fields in the theory. We will write down the conditions for the existence of moduli associated with the metric. To begin, recall that the metric satisfy the Hermitian condition

$$g_{mn} = J_m{}^r J_n{}^s g_{rs}. \tag{2.23}$$

In varying the metric, the Hermitian condition can require a corresponding variation in the complex structure. However, the complex structure cannot be deformed arbitrarily [29]. Starting from the standard complex structure, $J_a{}^b = i\delta_a{}^b$ and $J_{\bar{a}}{}^{\bar{b}} = -i\delta_{\bar{a}}{}^{\bar{b}}$, consider the deformation

$$J_m{}^n \to J'_m{}^n = J_m{}^n + \tau_m{}^n. \tag{2.24}$$

The condition $J'_m{}^r J'_r{}^n = -\delta_m{}^n$ implies that the only non-zero components are $\tau_{\bar{a}}{}^b$ and their complex conjugates $\tau_a{}^{\bar{b}}$. Furthermore, imposing the vanishing of the Nijenhuis tensor leads to the requirement that $\tau_{\bar{a}}{}^b$ are elements of $H^1(T)$, the first Dolbeault cohomology of the internal manifold $K$ with values in the holomorphic tangent bundle $T$. With the $SU(3)$ holonomy, we can use the holomorphic three-from, $w_{abc}$, to construct the associated $(2, 1)$-forms, $\chi_{\bar{a}bc} = \tau_{\bar{a}}{}^a w_{abc}$, which are $\bar{\partial}$ closed and elements of $H^{(2,1)}$. Therefore, the dimension of the complex structure deformation on any complex manifold with $SU(3)$ holonomy is given by the Hodge number $h^{(2,1)}$.

Now varying the Hermitian condition (2.23) to linear order, we find that variations of the complex structure, $\delta J_m{}^n = \tau_m{}^n$ with $\tau_{\bar{a}}{}^b \in H^1(T)$, and the metric variations of pure components are linked as follows

$$\delta g_{ab} = \frac{i}{2}\big(\tau_a{}^{\bar{c}} g_{b\bar{c}} + \tau_b{}^{\bar{c}} g_{a\bar{c}}\big), \tag{2.25}$$

and similarly for the complex conjugates $\delta g_{\bar{a}\bar{b}}$.[7] We can therefore divide metric variations into two types. The pure components, $\delta g_{ab}$ and $\delta g_{\bar{a}\bar{b}}$, are associated with the deformations of the

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[7] Since all fields are real, we will not explicitly write out components of fields which are just complex conjugates of other components.

complex structure, while the mixed components, $\delta g_{a\bar{b}}$, are independent of the complex structure. The mixed components however are associated with the variation of the Hermitian form. With $J_{mn} = J_m{}^k g_{kn}$, the variation of the Hermitian form is given by

$$\delta J_{ab} = \tau_a{}^{\bar{c}} g_{b\bar{c}} + i\delta g_{ab} = \frac{1}{2}\big(\tau_a{}^{\bar{c}} g_{b\bar{c}} - \tau_b{}^{\bar{c}} g_{a\bar{c}}\big),$$
$$\delta J_{a\bar{b}} = i\delta g_{a\bar{b}}. \tag{2.26}$$

We now perform a linearized variation on the supersymmetry constraint equations. A variation in the metric will generically require a variation in both the H-flux and the dilaton. This can be seen in the variation of the two constraint equations, (2.8) and (2.9), which we rewrite here in variational form

$$\delta H_{mnp} = 3\delta\big(J_{[m}{}^r \nabla_{|r|} J_{np]}\big), \tag{2.27}$$

$$(d\delta\phi)_m = \frac{1}{4}\delta\big(J_m{}^n H_{nrs} J^{rs}\big). \tag{2.28}$$

Recalling that $H = i(\bar{\partial} - \partial)J$, we see that the right-hand side of both equations above depend only on the metric and its variation. Therefore, (2.27) and (2.28) are the determining equations for the variations of the H-flux and the dilaton, respectively. Conversely, any condition on the variations of the H-flux or the dilaton will reflect back as a constraint for the metric variation. From the form of Eq. (2.28), a constraint for metric variations is immediately apparent. We see that a solution for $\delta\phi$ exists if and only if the one-form $A_m = \delta(J_m{}^n H_{nrs} J^{rs})$ is exact. (If we treat $A_m$ as a $U(1)$ gauge field, then we have the pure gauge condition, $A = d\Lambda$.) The trivial integrability condition $d^2\delta\phi = 0$ gives a non-trivial metric variational constraint

$$(dA)_{mn} = 2\partial_{[m}\delta\big(J_{n]}{}^p H_{prs} J^{rs}\big) = 0. \tag{2.29}$$

Now more explicitly, (2.27) gives the H-flux variation

$$\delta H_{abc} = \frac{i}{2}\big(\tau_a{}^{\bar{d}} H_{\bar{d}bc} + \tau_b{}^{\bar{d}} H_{a\bar{d}c} + \tau_c{}^{\bar{d}} H_{ab\bar{d}}\big), \tag{2.30}$$

$$\delta H_{ab\bar{c}} = -3i\,\partial_{[a}\delta J_{b\bar{c}]} + i\big(\tau_a{}^{\bar{d}} H_{\bar{d}b\bar{c}} + \tau_b{}^{\bar{d}} H_{a\bar{d}\bar{c}}\big). \tag{2.31}$$

The equation for $\delta H_{abc}$ can also be obtained by requiring that the $(3,0)$ component of $H$ with respect to the deformed complex structure remains zero. And also, we obtain from (2.28)

$$\partial_a\delta\phi = \frac{1}{2}\tau_a{}^{\bar{d}} H_{\bar{d}b\bar{c}} J^{b\bar{c}} + \frac{i}{4}\big(2H_{ab\bar{c}}\delta J^{b\bar{c}} + 2\delta H_{ab\bar{c}} J^{b\bar{c}} + H_{a\bar{b}\bar{c}}\delta J^{\bar{b}\bar{c}}\big) \tag{2.32}$$

with $\delta J^{\bar{b}\bar{c}} = \frac{1}{2}(g^{\bar{b}d}\tau_d{}^{\bar{c}} - g^{\bar{c}d}\tau_d{}^{\bar{b}})$. The above three equations, (2.30), (2.31), and (2.32), represent necessary variational conditions relating the metric, H-flux, and the dilaton.

The above variational conditions can easily reproduce the metric moduli space for the zero-flux Calabi–Yau solution. With $H = 0$, the variational equations for $\delta H$ and $\delta\phi$ above trivialize and become independent of $\tau_m{}^n$. Therefore, each of the $h^{(2,1)}$ complex structure deformations with $\tau_{\bar{a}}{}^b \in H^1(T)$ represents a modulus. Furthermore, requiring $\delta H = 0$ so that the solution remains Calabi–Yau, (2.31) implies that $d(\delta J) = 0$, which is a variation on the Kähler condition, $dJ = 0$, and has $h^{(1,1)}$ independent solutions. (Those $\delta J$ which are exact correspond to diffeomorphism and are thus modded out.) Altogether for the zero-flux Calabi–Yau, we have $h^{(2,1)} + h^{(1,1)}$ independent metric variations that do not involve either the H-flux or the dilaton field.

When the H-flux is non-zero, characterizing all metric variations that satisfy even just the metric variational constraint (2.29) seems difficult. To proceed further, we will make a simplifying restriction by taking the H-flux to be fixed, or equivalently, imposing $\delta H = 0$. Fixing the H-flux will typically also fix the complex structure. From (2.30), we have the constraint on complex structure that

$$\delta H_{abc} = \frac{i}{2}\left(\tau_a{}^{\bar{d}}H_{\bar{d}bc} + \tau_b{}^{\bar{d}}H_{a\bar{d}c} + \tau_c{}^{\bar{d}}H_{ab\bar{d}}\right) = 0. \tag{2.33}$$

This can be a strong constraint since both $\tau_m{}^n$ and $H$ are generically functions and are directly dependent on the geometry. Moreover, if there are regions of the manifold where $dH = 0$ (as is the case for the orbifold background we discuss in the next section), then the integral of $H$ over three-cycles in these regions must be quantized (2.5). Though the integral of $H$ is independent of the Hermitian metric, it can depend on the complex structure via the limits of integration for the coordinates. Specifically, the boundary conditions for the coordinates on a compact manifold can be modified by a deformation of the complex structure.[8] Hence, imposing the Dirac quantization in regions where $dH = 0$ can also fix the complex structure. Below, we shall analyze variations in a fixed H-flux background and assume that the complex structure has been fixed.

With $\delta H = \tau_m{}^n = 0$, the two conditions, (2.31) and (2.32), and their complex conjugates can be expressed as

$$\delta H = i(\bar{\partial} - \partial)\delta J = 0, \tag{2.34}$$

$$(d\delta\phi)_m = \frac{1}{4}J_m{}^n H_{nrs}\delta J^{rs}, \tag{2.35}$$

where $\delta J$ denotes the variation of the mixed components of the Hermitian form, $\delta J_{a\bar{b}}$. The two equations can also be obtained directly by varying equations (2.11) and (2.9). Interestingly, in a fixed H-flux background, though $dJ \neq 0$, (2.34) implies that $d(\delta J) = 0$ or that the variation of the Hermitian form is required to be a closed (1, 1)-form. Eq. (2.35) is the determining equation for the variation of the dilaton. The associated integrability condition gives the second constraint for $\delta J$

$$(dA)_{mn} = 2\partial_{[m}\left(J_{n]}{}^p H_{prs}\delta J^{rs}\right) = 0. \tag{2.36}$$

This second constraint is non-trivial and involves the fixed H-flux. A valid deformation of $\delta J$ must satisfy both of the above conditions, (2.34) and (2.36).

Together, the above two conditions for $\delta J$ are not easily satisfied. Consider first an exact variation, $\delta J = i\partial\bar{\partial}\Phi$. Being exact, it is trivially a closed (1, 1)-form. However, an exact variation of $J$ typically does not satisfy (2.36). (We will give an explicit example of this in the next section.) Therefore, although $\delta J$ is closed, the moduli are not related to the standard de Rham cohomology. As a second example, consider the radial modulus $\delta J_{a\bar{b}} = aJ_{a\bar{b}}$ which is present in Calabi–Yau compactifications and represents an overall rescaling of the internal metric. A variation of $\delta\phi = \phi/a$ will satisfy (2.35) (and hence also (2.36)) but now (2.34) is not satisfied. A radial modulus would give $\delta H = i(\bar{\partial} - \partial)\delta J = aH \neq 0$ which is not allowed. The essential point here is that $J$ is not a closed (1, 1)-form for non-Kähler flux backgrounds. We thus see that fixing the H-flux also fixes the overall scale of the metric.

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[8] Consider the simple example of a torus. The complex structure is parametrized by the modular parameter $\tau = \tau_1 + i\tau_2$. The metric can be written as $ds^2 = A\,dz\,d\bar{z}$ where $A$ is the area of the torus and $z = \frac{1}{\sqrt{\tau_2}}(x + \tau y)$ with $x \sim x + 1$ and $y \sim y + 1$. Clearly, the boundary conditions (or the periodicities) of $z$ depend on $\tau$.

## 3. Moduli of the heterotic solution dual to type IIB orientifold

The $N = 1$ heterotic solution with H-flux discussed in [12–14,30] is dual to type IIB $N = 1$ flux compactification on an orientifold. Without taking account of non-perturbative effects, the IIB orientifold models with fluxes lift all complex structure moduli but leave unfixed certain Kähler moduli [31,32]. This suggests at the supergravity level, the heterotic dual also contain moduli. In this section, we find the heterotic moduli via duality and show that they satisfy the conditions discussed in the previous section.

### 3.1. IIB orientifold model and its moduli

We first review the type IIB flux model on a toroidal orientifold described in [12–14]. The background geometry is $\text{K3} \times T^2/\mathbb{Z}_2$ where $\mathbb{Z}_2 = \Omega(-1)^{F_L}\mathcal{I}_2$ and $\mathcal{I}_2$ is the reflection operator reversing the two directions of $T^2$.[9] The solution originates from an M-theory (or dual F-theory) flux compactification on $\text{K3} \times \text{K3}$. Localized at each of the four fixed points of $T^2/\mathbb{Z}_2$ are four D7-branes and an O7-plane. For concreteness, the orbifold limit of K3 is taken, replacing K3 with $T^4/\mathcal{I}_4$. This allows us to write down a warped metric of the form

$$ds^2 = \Delta^{-1}\eta_{\mu\nu}\,dx^\mu\,dx^\nu + \Delta g_{mn}\,dy^m\,dy^n, \tag{3.1}$$

where $g_{mn}$ is the metric on $T^4/\mathcal{I}_4 \times T^2/\mathbb{Z}_2$ and the warped factor $\Delta$ is related to the five-form field strength

$$F_5 = dD_4 - \frac{1}{2}C_2 \wedge H_3 + \frac{1}{2}B_2 \wedge F_3 = (1 + *_{10})\big(d\Delta^2 \wedge dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3\big). \tag{3.2}$$

Here, the Hodge star, $*_{10}$, is defined with respect to the ten-dimensional metric. The self-duality of $F_5$ further implies

$$*_6 d\Delta^2 = -\frac{1}{2}C_2 \wedge H_3 + \frac{1}{2}B_2 \wedge F_3, \tag{3.3}$$

where $*_6$ is defined with respect to the unwarped six-dimensional metric, $g_{mn}$ in (3.1). This condition relates the warp factor to the internal metric and the fluxes.

In order to preserve $N = 1$ supersymmetry, the three-form flux, $G_3$, is required to be a primitive (2, 1)-form [11]. For a diagonal metric with $J = ig_{1\bar{1}}\,dz_1 \wedge d\bar{z}_1 + ig_{2\bar{2}}\,dz_2 \wedge d\bar{z}_2 + ig_{3\bar{3}}\,dz_3 \wedge d\bar{z}_3$, the constant non-localized part takes the form[10]

$$\begin{aligned} G_3 &= F_3 - \tau H_3 \\ &= -2i\,\text{Im}[\tau]\big(\bar{A}\,dz_1 \wedge d\bar{z}_2 \wedge dz_3 + \bar{B}\,d\bar{z}_1 \wedge dz_2 \wedge dz_3 + \bar{C}\,dz_1 \wedge dz_3 \wedge d\bar{z}_3\big) \end{aligned} \tag{3.4}$$

with $\tau = C_0 + ie^{-\phi_B}$ and

$$\begin{aligned} H_3 &= A\,d\bar{z}_1 \wedge dz_2 \wedge d\bar{z}_3 + B\,dz_1 \wedge d\bar{z}_2 \wedge d\bar{z}_3 + C\,d\bar{z}_1 \wedge d\bar{z}_2 \wedge dz_3 + \text{c.c.}, \\ F_3 &= \tau(A\,d\bar{z}_1 \wedge dz_2 \wedge d\bar{z}_3 + B\,dz_1 \wedge d\bar{z}_2 \wedge d\bar{z}_3 + C\,d\bar{z}_1 \wedge d\bar{z}_2 \wedge dz_3) + \text{c.c.}, \end{aligned} \tag{3.5}$$

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[9] The $\Omega$ operator here refers to the worldsheet parity operator and should not be confused with the holomorphic three-form in Section 2.

[10] Additionally, $G_3$ can have a metric dependent term proportional to $(g_{1\bar{1}}\,dz_1 \wedge d\bar{z}_1 - g_{2\bar{2}}\,dz_2 \wedge d\bar{z}_2) \wedge dz_3$ which we do not consider here for simplicity.

where $A$, $B$ and $C$ are complex constants and "c.c." denotes the complex conjugates. Above, the $z_i$'s are the holomorphic coordinates of the internal six-manifold. We will choose the associated two-form potentials to be

$$B_2 = \left[\left(A\bar{z}_1 + \frac{\bar{C}}{2}z_1\right)dz_2 - \left(B\bar{z}_2 + \frac{\bar{C}}{2}z_2\right)dz_1\right] \wedge d\bar{z}_3 + \text{c.c.},$$

$$C_2 = \left[\left(\tau A\bar{z}_1 + \bar{\tau}\frac{\bar{C}}{2}z_1\right)dz_2 - \left(\tau B\bar{z}_2 + \bar{\tau}\frac{\bar{C}}{2}z_2\right)dz_1\right] \wedge d\bar{z}_3 + \text{c.c.} \tag{3.6}$$

Additionally, the three-form fluxes are constrained by the D3-brane tadpole cancellation condition

$$\frac{\chi(K3 \times K3)}{24} = -\int_K H_3 \wedge F_3 + N_{D3}, \tag{3.7}$$

where $N_{D3}$ is the number of D3-branes present and we have set $(2\pi)^2\alpha' = 1$ in line with [13,14]. The three-form fluxes are also quantized,

$$\int_\Gamma H_3 \in \mathbb{Z}, \qquad \int_\Gamma F_3 \in \mathbb{Z}, \tag{3.8}$$

where the integration is over any three-cycles.

We now consider the moduli of this type IIB orientifold model. We start with the complex structure moduli. The condition that $G_3$ is a $(2,1)$-form can either fix the complex structure moduli and $\tau$ when a set flux is turned on, or determine the allowed flux for a given complex structure and $\tau$. For ease of performing duality, it is more useful to set the complex structure by defining the holomorphic variables $z_1 = y_4 + iy_5$, $z_2 = y_6 + iy_7$, $z_3 = y_8 + iy_9$ (with the real coordinates having unit periodicity on the covering space) and also set

$$\tau = C_0 + ie^{-\phi_B} = i. \tag{3.9}$$

The additional constraints on the fluxes, (3.7) and (3.8), then become

$$24 = 4\left(|A|^2 + |C|^2 + |B|^2\right) + N_{D3}, \tag{3.10}$$

$$\text{Re}\,A \pm \text{Re}\,B \pm \text{Re}\,C \in 2\mathbb{Z}, \qquad \text{Im}\,A \pm \text{Im}\,B \pm \text{Im}\,C \in 2\mathbb{Z}. \tag{3.11}$$

These fix the fluxes to discrete values of $(A, B, C)$. Some examples with $N_{D3} = 0$ are for instance $(A, B, C) = (2 + i, i, 0)$ [12,13], $(A, B, C) = (2, 1, 1)$ [14] and also $(A, B, C) = (1, 1, 2)$ where $A = B$.

With fixed complex structure and axion/dilaton, we turn now to the Kähler moduli. Recall that the Kähler moduli arise from varying the Kähler form $J \to J + \delta J$ which changes the metric via the relation $J_{a\bar{b}} = ig_{a\bar{b}}$. When $G_3 = 0$, the number of Kähler moduli is given by $h^{1,1}$. For a nonzero fixed $G_3$, the supersymmetry condition that $G_3$ is primitive will lift some of the Kähler moduli [31]. For a middle-dimensional form such as $G_3$ (i.e., a 3-form in six dimensions), the primitivity condition is equivalent to the constraint $G_3 \wedge J = 0$. The unlifted Kähler moduli are those that satisfy

$$G_3 \wedge \delta J = 0. \tag{3.12}$$

This ensures that $G_3$ remains primitive and its self-duality property $*G_3 = -iG_3$ unchanged with respect to the deformed metric.[11] For generic values of $(A, B, C)$, the Kähler deformations that satisfy (3.12) are

$$\delta J = i\delta k_1 \, dz_1 \wedge d\bar{z}_1 + i\delta k_2 \, dz_2 \wedge d\bar{z}_2 + i\delta k_3 \, dz_3 \wedge d\bar{z}_3. \tag{3.13}$$

For special values of $(A, B, C)$ such that $A = \pm B$, an additional deformation is present. Consider, for example, $(A, B, C) = (1, 1, 2)$, the deformation of

$$\delta J = i\delta k_4 (dz_1 \wedge d\bar{z}_2 + dz_2 \wedge d\bar{z}_1) \tag{3.14}$$

is also allowed. Note that as required, these Kähler deformations are real and invariant under the orientifold action.

We will incorporate the Kähler deformations directly into the internal six-dimensional metric. We promote the deformation variables $\delta k_i$ into parameters of the metric and write the metric as follows

$$ds^2 = \Delta g_{mn} \, dy^m \, dy^n = 2\Delta g_{a\bar{b}} \, dz^a \, dz^{\bar{b}} \tag{3.15}$$

with

$$g_{a\bar{b}} = \begin{pmatrix} k_1 & k_4 & 0 \\ k_4 & k_2 & 0 \\ 0 & 0 & k_3 \end{pmatrix}. \tag{3.16}$$

The $k_i$'s, for $i = 1, 2, 3$, parametrizes the volume of each torus. And $k_4$ which is present only if $A = B$, mixes the two tori of $T^4/\mathcal{I}_4$.[12] With explicit expressions for both the metric and 3-form fluxes, we can now determine the warp factor using the condition for the self-duality of $F_5$ in (3.3). The warp factor satisfies two independent differential equations

$$k_3(k_2\partial_{\bar{z}_1} - k_4\partial_{\bar{z}_2})\Delta^2 = -\left(|A|^2 + \frac{1}{2}|C|^2\right)z_1, \tag{3.17}$$

$$k_3(k_1\partial_{\bar{z}_2} - k_4\partial_{\bar{z}_1})\Delta^2 = -\left(|B|^2 + \frac{1}{2}|C|^2\right)z_2, \tag{3.18}$$

which has the solution

$$k_3\Delta^2 = c_0 - \frac{1}{k_1 k_2 - k_4^2}\left\{ k_1\left(|A|^2 + \frac{1}{2}|C|^2\right)|z_1|^2 + k_2\left(|B|^2 + \frac{1}{2}|C|^2\right)|z_2|^2 \right.$$
$$\left. + \frac{1}{2}k_4\left(|A|^2 + |B|^2 + |C|^2\right)(z_1\bar{z}_2 + z_2\bar{z}_1) \right\}, \tag{3.19}$$

where $c_0$ is an integration constant which ensures that $\Delta^2$ is positive definite. It is worthwhile to point out that the warp factor has dependence on the Kähler moduli. This has also been noted in [33].

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[11] In our convention for the Hodge star, the equation of motion requires $G_3$ to be imaginary anti-self-dual. Note that all deformations satisfying (3.12) will preserve the imaginary anti-self-duality condition of $G_3$ since any primitive $(p, q)$-form for $p + q = 3$ satisfies $*A^{(p,q)} = i(-1)^q A^{(p,q)}$. Also, as noted in [33], those Kähler moduli which do not satisfy (3.12) may still correspond to flat directions if we allow $G_3$ to vary. However, here $G_3$ is kept fixed so those moduli not satisfying (3.12) are lifted.

[12] We have included the $k_4$ parameter to illustrate that any deformation that arises for special values of $(A, B, C)$ can be incorporated into the duality mapping performed below.

We have obtained the unlifted Kähler moduli and will proceed to dualize the moduli in the next subsection. But before doing so, it is important to point out that we have thus far ignored the contributions from the localized fixed points. The localized fixed points generate singularities for the warp factor. They also allow for localized $G_3$ fluxes corresponding to gauge fluxes on the D7-branes. These two issues do play an important role in the consistency of the theory and have been worked out in detailed in [12,13]. Moreover, if we treat $T^4/\mathcal{I}_4$ as a true K3 surface, then we should also consider the metric deformations at the orbifold fixed points. To see that they are present, recall first that K3 has 58 metric deformations [34]. Of these, only 10 are due to the deformation of the $T^4$ metric. The rest, $48 = 3 \times 16$, comes from the 16 fixed points which must be blown up and each replaced with an Eguchi–Hanson space. Of the three metric deformations of the Eguchi–Hanson space, one is an overall scaling and two are rotations [35]. Some of the moduli at the fixed points can be fixed by the localized $G_3$ flux but not by the non-localized $G_3$ flux in (3.4). However, since the duality mapping that we will perform in the next subsection is only at the level of supergravity, we will focus only on the non-localized moduli and not consider further the localized moduli at the singularities.

### 3.2. Dual heterotic model and its moduli

Without fluxes, the type IIB orientifold model on K3 $\times$ $T^2/\mathbb{Z}_2$ becomes the type IIB theory on K3 $\times$ $T^2/\Omega$ after applying T-duality in both directions of $T^2$. This is equivalent to the type I theory on K3 $\times$ $T^2$, which in turn is S-dual to the $SO(32)$ heterotic theory on K3 $\times$ $T^2$. The same duality chasing can be worked out with the presence of fluxes for the IIB orientifold model on $T^4/\mathcal{I}_4 \times T^2/\mathbb{Z}_2$. Note that all type IIB fields in the previous subsection do not have any dependence on $z_3$, $\bar{z}_3$, which are the directions which we T-dualize. The transformation rules of supergravity fields under T- and S-dualities are derived for example in [36–38]. Applying the dualities, the heterotic dual has a $M^{3,1} \times K$ background geometry, and the dilaton, internal metric, and $B$-field are given as follows[13]

$$e^\phi = 2\Delta k_3, \tag{3.20}$$

$$ds^2 = \frac{e^{2\phi}}{k_3}\big(k_1\,dz_1\,d\bar{z}_1 + k_4(dz_1\,d\bar{z}_2 + dz_2\,d\bar{z}_1) + k_2\,dz_2\,d\bar{z}_2\big)$$
$$+ \big|dz_3 + 2(B_2)_{a\bar{z}_3}\,dz^a\big|^2, \tag{3.21}$$

$$B_{mn} = i(B_2 \wedge C_2)_{mnz_3\bar{z}_3}, \qquad B_{mz_3} = -i(C_2)_{mz_3}, \qquad B_{m\bar{z}_3} = i(C_2)_{m\bar{z}_3}. \tag{3.22}$$

Substituting the expressions for $B_2$ and $C_2$ in (3.6), we have more explicitly,

$$g_{a\bar{b}} = \frac{1}{2}\begin{pmatrix} e^{2\phi}\frac{k_1}{k_3} + |2B\bar{z}_2 + \bar{C}z_2|^2 & e^{2\phi}\frac{k_4}{k_3} - (2B\bar{z}_2 + \bar{C}z_2)(2\bar{A}z_1 + C\bar{z}_1) & -2B\bar{z}_2 - \bar{C}z_2 \\ e^{2\phi}\frac{k_4}{k_3} - (2\bar{B}z_2 + C\bar{z}_2)(2A\bar{z}_1 + \bar{C}z_1) & e^{2\phi}\frac{k_2}{k_3} + |2A\bar{z}_1 + \bar{C}z_1|^2 & 2A\bar{z}_1 + \bar{C}z_1 \\ -2\bar{B}z_2 - C\bar{z}_2 & 2\bar{A}z_1 + C\bar{z}_1 & 1 \end{pmatrix}, \tag{3.23}$$

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[13] Except for the type IIB potentials, $B_2$ and $C_2$, all other background fields in this subsection are heterotic fields.

$$B = \left(\bar{B}\bar{C}z_2^2 - BC\bar{z}_2^2\right) dz_1 \wedge d\bar{z}_1 + \left(\bar{A}\bar{C}z_1^2 - AC\bar{z}_1^2\right) dz_2 \wedge d\bar{z}_2$$
$$+ \Big[ (BC\bar{z}_1\bar{z}_2 - \bar{A}\bar{C}z_1z_2) \, dz_1 \wedge d\bar{z}_2$$
$$+ \left(\left(\bar{B}z_2 - \frac{C}{2}\bar{z}_2\right) d\bar{z}_1 - \left(\bar{A}z_1 - \frac{C}{2}\bar{z}_1\right) d\bar{z}_2\right) \wedge dz_3 + \text{c.c.} \Big]. \tag{3.24}$$

We see from (3.21) that the internal space consists of a warped K3 (or more precisely $T^4/\mathcal{I}_4$) with a non-trivial $T^2$ fiber. The twisting of the $T^2$ fiber arises from the two T-duality transformations and is due solely to the presence of the $B$-field in the IIB orientifold model [39]. Further insight can be obtained by decomposing this six-dimensional compactification as a four-dimensional compactification plus a two-dimensional one. First, in compactifying down to six dimensions, 6d $N = 1$ supersymmetry requires that the compact four-manifold be a conformal Calabi–Yau with the warp factor $e^{2\phi}$ [19,40]. Here, we have a conformal K3 with exactly the warping mandated by supersymmetry. Now if we proceed further to compactify on a $T^2$, we will obtain a background with 4d $N = 2$ supersymmetry. Thus, the presence of the non-trivial $T^2$ fibering here is required to break the additional supersymmetries so that we end up with the desired $N = 1$ supersymmetry in four dimensions.

The dual heterotic background can be checked to satisfy the heterotic supersymmetry constraints (2.11) and (2.9). The metric (3.23) is in fact conformally balanced. Using $H = i(\bar{\partial} - \partial)J$, the non-zero $H$-field components (modulo complex conjugation) are

$$H_{\bar{1}\bar{2}2} = 3AC\bar{z}_1, \qquad H_{\bar{1}1\bar{2}} = 3BC\bar{z}_2, \qquad H_{13\bar{2}} = \bar{A},$$
$$H_{32\bar{1}} = \bar{B}, \qquad H_{12\bar{3}} = \bar{C}. \tag{3.25}$$

We note that the contributions to the $H$-field come from both the warp factor and the $T^2$ fiber part of the metric. We have however used the warp factor relations, (3.17) and (3.18), to cancel off additional terms in the first two components above. The $H$ values agree with those obtained from $H = dB$ using (3.24). And for $C \neq 0$, $H_{\bar{1}\bar{2}2}$ and $H_{\bar{1}1\bar{2}}$, like the $B$-field, are only locally defined on patches. Moreover, it is important to point out that we have $dH = 0$ here only because the contributions localized at the fixed points of the background geometry have not been taken into account. The curvature singularities at the fixed points of $T^4/\mathcal{I}_4$ lead to additional contributions for the $H$-field through the warp factor [12]. Indeed, from a no-go theorem, $dH$ cannot vanish in a non-zero flux background [41,42].

The moduli of the heterotic model can be simply obtained by varying the $k_i$'s in the expressions for the background fields in (3.20), (3.21), and (3.25). We find that the variations only deform the dilaton field and the conformal $T^4/\mathcal{I}_4$ part of the metric. The H-flux given in (3.25) has no dependence on the $k_i$'s and is thus fixed. The complex structure of the heterotic model is also fixed in the fixed H-flux background.

Let us now consider the variations in more detail. Take first the $k_3$ variation. In the IIB orientifold model, $k_3$ varies the volume of $T^2/\mathbb{Z}_2$. In contrast, the heterotic dual metric has no dependence on $k_3$ at all. The factor $e^{2\phi}/k_3 = 4k_3\Delta^2$ in the metric (3.21), as given explicitly in (3.19), is $k_3$ invariant. $k_3$ only shifts the dilaton and is thus the heterotic dilaton modulus. As for $k_1, k_2, k_4$, the remaining non-localized Kähler moduli of $T^4/\mathcal{I}_4$ in the IIB orientifold model, we find that their variations in the heterotic model deform both $\phi$ and $g_{a\bar{b}}$. Take for example $k_1$. Its variation results in the following field deformations

$$\delta_{k_1}\phi = \frac{1}{2k_1^2} \frac{(|B|^2 + \frac{1}{2}|C|^2)|z_2|^2}{k_3\Delta^2}\delta k_1, \tag{3.26}$$

$$\delta_{k_1} g_{a\bar{b}} = \begin{pmatrix} 2(c_0 - \frac{1}{k_2}(|A|^2 + \frac{1}{2}|C|^2)|z_1|^2) & 0 & 0 \\ 0 & 2\frac{k_2}{k_1^2}(|B|^2 + \frac{1}{2}|C|^2)|z_2|^2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \delta k_1, \qquad (3.27)$$

where we have set $k_4 = 0$ above to simplify the expressions.[14] The non-zero variation for $g_{2\bar{2}}$ comes from the $k_1$ dependence of the warp factor. ($\delta g_{1\bar{2}}$ and $\delta g_{2\bar{1}}$ are also non-zero if $k_4 \neq 0$.) As is evident, the simple Kähler moduli in the IIB orientifold model have been mapped to nontrivial deformations in the heterotic dual. Nevertheless, the deformations can be straightforwardly checked to satisfy both $d(\delta J) = 0$ (2.34) and the determining equation for $\delta \phi$ in (2.35). Thus we have via duality provided explicit variations that satisfy the two moduli conditions given in Section 2.2 for a fixed H-flux background.

The explicit metric moduli that we have found are in fact very special. Indeed, it is not easy to write down metric variations that satisfy both variational constraints. Even for variations $\delta J$ that are exact and hence trivially closed, the second condition, requiring the 1-form $A_m = J_m{}^n H_{nrs} \delta J^{rs}$ also be exact, is not automatically satisfied. We demonstrate this with an explicit example. For simplicity, we take $k_1 = k_2 = k_3 = 1$ and $k_4 = C = 0$. To construct the exact 2-form, we utilize the natural 1-form on the $T^2$ fiber, $\alpha = dz_3 + 2(B_2)_{a\bar{z}_3} dz^a = dz_3 + 2A\bar{z}_1 dz_2 - 2B\bar{z}_2 dz_1$. We take

$$\delta J = \epsilon\, d(\alpha + \bar{\alpha}) = 2\epsilon(\bar{A} + B)\, dz_1 \wedge d\bar{z}_2 - 2\epsilon(A + \bar{B})\, dz_2 \wedge d\bar{z}_1, \qquad (3.28)$$

where $\epsilon$ is the infinitesimal parameter. The values of $A_m$ can now be straightforwardly worked out with the H-flux given in (3.25). We find

$$A_{z_1} = -8i\epsilon e^{-4\phi} \bar{A} B (A + \bar{B})\bar{z}_2,$$
$$A_{z_2} = 8i\epsilon e^{-4\phi} A \bar{B} (\bar{A} + B)\bar{z}_1,$$
$$A_{z_3} = 4i\epsilon e^{-4\phi}(|A|^2 + |B|^2 + 2\bar{A}\bar{B}). \qquad (3.29)$$

Clearly, $A_m$ has no dependence on $z_3$ and $\bar{z}_3$ since no fields in the heterotic dual model has dependence in the coordinates which were T-dualized. But as can be easily checked, $\partial_{z_1} A_{z_3}$ and $\partial_{z_2} A_{z_3}$ are non-zero, and therefore, $(dA)_{z_1 z_3}$ and $(dA)_{z_2 z_3}$ do not vanish. We thus conclude that $A_m$ is not exact and $\delta J$ as given in (3.28) is not a valid deformation. Hence, we have shown that a variation of $\delta J$ by an exact form does not generically give a new solution.

Finally, it is significant that the $T^2$ fiber is not affected by any of the $k_i$ variations. With $2\pi\sqrt{\alpha'} = 1$, the area of the $T^2$ fiber is fixed and is of order $\alpha'$ [5,30]. Therefore, there is no overall radial modulus in the heterotic model, agreeing with our analysis in Section 2.2. This is as expected since the heterotic dual backgrounds cannot have a continuous deformation to a large radius Calabi–Yau geometry [13]. However, the small size of the $T^2$ fiber may point to the need for incorporating $\alpha'$ corrections into the analysis. But with regard to the metric moduli that we have found via duality, it is useful to emphasize that these moduli only involve the conformal $T^4$ base and does not affect the small directions of the geometry. The K3 base can be taken to be sufficiently large such that the supergravity analysis on the conformal $T^4/\mathcal{I}_4$ remain valid. A large $T^4/\mathcal{I}_4$ base is in fact also required in order to neglect the contributions from the fixed-point singularities. Heuristically, we can think of the fixed points as located at the "boundaries" of $T^4/\mathcal{I}_4$ and we have studied the moduli in the "bulk", far away from the fixed points.

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[14] Recall that the $k_4$ deformation is present only if $A = B$. A non-zero $k_4$ will add more terms to the expressions.

## 4. Discussion

We have analyzed the linearized variational equations of the supersymmetry constraint equations to study moduli associated with the metrics. In a fixed H-flux background and with the complex structure fixed, we wrote down two conditions (2.34) and (2.36), which metric variations must satisfy. Our analysis in the heterotic theory is also directly applicable to the type I theory and to the subsector of type II theories where only the H-flux is turned on.

Our focus on the metric moduli in a fixed H-flux background is primarily motivated by a desire to understand the heterotic dual of the unlifted Kähler moduli in certain type IIB orientifold models. However, in order to understand the full heterotic moduli space, it is necessary as the next step to consider more general deformations. Generically, we expect there can be moduli associated with the variation of the complex structure if the H-flux is allowed to vary. Another source of moduli can come from the gauge fields. From the gaugino supersymmetry transformation, the gauge field strengths $F_{mn}$ are required to satisfy the Hermitian Yang–Mills equations, i.e., $F_{ab} = F_{\bar{a}\bar{b}} = 0$ and $g^{a\bar{b}} F_{a\bar{b}} = 0$. To our knowledge, not much is known about the space of gauge field solutions in conformally balanced geometries. (Recently, progress in constructing non-Abelian gauge field configurations in non-Kähler geometries has appeared in [43].) Note that the gauge field strengths, the metric, and the H-flux are all connected via the modified Bianchi identity (2.4). Thus, it would be interesting to understand if a variation in the gauge field can be independent or would also require a variation in the metric and the H-flux. And conversely, whether a metric variation that lead to a variation of $\text{tr}(R^{(-)} \wedge R^{(-)})$ can always be accompanied by a compensating variation in $\text{tr}(F \wedge F)$ in backgrounds with $dH \neq 0$. These issues are important and worth pursuing.

Nevertheless, knowing certain required conditions for the metric moduli in a fixed flux background, it is interesting to ask whether there is a simple method to count the number of moduli satisfying these conditions. For instance, on a Calabi–Yau, the equation $d(\delta J) = 0$ is the condition for the Kähler moduli and the number of moduli is given simply by the Hodge number $h^{(1,1)}$. So perhaps the number of metric moduli can also be simply expressed in terms of the dimension of a generalized cohomology. We briefly explore this issue below [44].

Consider first the IIB case. The unlifted Kähler moduli must satisfy the two constraints

$$d\delta J = 0, \qquad G_3 \wedge \delta J = 0. \tag{4.1}$$

Notice that the two constraints each separately defines a cohomology, i.e., $d^2 = 0$ and also $G \wedge G \wedge = 0$. But since we are interested in constructing one cohomology, consider combining the two together as

$$\mathcal{D}\delta J = (d + G_3 \wedge)\delta J = 0. \tag{4.2}$$

The differential operator $\mathcal{D} = d + G_3 \wedge$ maps the two-form $\delta J$ into a direct sum of a three-form and a five-form. More generally, it maps the space of even forms to odd forms and vice verser. These types of operators have appeared in flux compactifications [16] expressed in the pure spinor formalism of Hitchin [15]. The three-form wedge $G_3 \wedge$ is called a twist and we will call $\mathcal{D}$ a twisted differential operator.[15] Now, if $dG_3 = 0$, as is the case when the axion/dilaton $\tau$ is a constant, then it can be easily checked that $\mathcal{D}\mathcal{D} = 0$. We can therefore treat the twisted

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[15] We note that $G_3$ is complex so the twisted differential operator is also complex. Also, we can consider replacing $\delta J$ with a pure spinor of the form $\delta(e^{iJ})$.

differential operator as defining a generalized cohomology. Note that only a one-form can be mapped into a two-form by the exterior derivative $d$ so $\delta J$ would still be modded out by an exact two-form. Therefore, an interesting question is whether the dimension of the generalized cohomology may be related to the dimension of the moduli.

As for the heterotic case, constructing a generalized cohomology is unfortunately not as straightforward. The two constraints are

$$d\delta J = 0, \qquad -2J_m{}^n(d\delta\phi)_n = \frac{1}{2}H_{mnp}\delta J^{np}. \tag{4.3}$$

The constraints again can be combined together. But first, we introduce the following definitions:

$$d^c = i(\bar{\partial} - \partial), \tag{4.4}$$

$$H \llcorner J = \frac{1}{2}\left(H_{mnp}g^{nr}g^{ps}J_{rs}\right)dy^m. \tag{4.5}$$

The $d^c$ operator is just $-J_m{}^n\partial_n$ in the standard complex structure where $J_a{}^b = i\delta_a{}^b$. This operator is natural in the heterotic theory since $dH \neq 0$ but we have

$$H = d^c J, \qquad d^c H = \left(d^c\right)^2 J = 0. \tag{4.6}$$

Further, we note that the H-contraction operator $H\llcorner$ can also be defined more generally to map a $p$-form into a $(p-1)$-form (for $p \geqslant 2$). The two constraints in (4.3) can now be re-written as

$$d^c\delta J = 0, \qquad 2d^c(\delta\phi) = -H \llcorner \delta J, \tag{4.7}$$

where the minus sign is due to $H_{mnp}\delta J^{np} = -H_{mnp}g^{nr}g^{ps}\delta J_{rs}$. Notice that the integrability condition of the second constraint equation given in (2.36) can be simply expressed as $d^c(H \llcorner \delta J) = 0$. We now formally combine the two constraints as follows.

$$\mathcal{D}\left(e^{2\delta\phi}e^{\delta J}\right) = \left(d^c + H\llcorner\right)\left(e^{2\delta\phi}e^{\delta J}\right) = 0, \tag{4.8}$$

where for $e^{2\delta\phi}e^{\delta J}$, only the linear order in variation should be kept, i.e., $e^{2\delta\phi}e^{\delta J} = 1 + 2\delta\phi + \delta J$. Here again, the $\mathcal{D}$ operator map even forms to odd forms and vice versa, but it is no longer the case that $\mathcal{DD} = 0$. Nevertheless, it is interesting to point out that if we had in the IIB theory wrote the primitivity constraint as $G_3 \llcorner \delta J = 0$, then the IIB and heterotic twisted differential operators would be similar except for exchanging $d \leftrightarrow d^c$ and $G_3 \leftrightarrow H_3$.

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