TOTALLY BOUNDED ELEMENTS IN W*-PROBABILITY SPACES

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Abstract. We introduce the notion of a totally (K-) bounded element of a W*-probability space (M, φ) and, borrowing ideas of Kadison, give an intrinsic characterization of the *-subalgebra M_{tb} of totally bounded elements. Namely, we show that M_{tb} is the unique strongly dense $*$-subalgebra M_0 of totally bounded elements of M for which the collection of totally 1-bounded elements of M_0 is complete with respect to the $\|\cdot\|_\varphi^\#$-norm and for which M_0 is closed under all operators $h_a(\log(\Delta))$ for $a \in \mathbb{N}$, where Δ is the modular operator and $h_a(t) := 1/\cosh(t-a)$ (see Theorem 4.3). As an application, we combine this characterization with Rieffel and Van Daele's bounded approach to modular theory to arrive at a new language and axiomatization of W*-probability spaces as metric structures. Previous work of Dabrowski had axiomatized W*-probability spaces using a smeared version of multiplication, but the subalgebra M_{tb} allows us to give an axiomatization in terms of the original algebra operations. Finally, we prove the (non-)axiomatizability of several classes of W*-probability spaces.

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1. Introduction

First introduced in earnest in the papers [6] and [7], the model theory of finite von Neumann algebras has been a very active area of research; see [8] for a survey of many of the results that have been obtained to date. In order to be treated as a metric structure, one must equip a finite von Neumann algebra M with a faithful, normal trace τ; such a tracial von Neumann algebra (or tracial W*-probability space) is then treated as a structure in a many-sorted language, where the sorts are operator norm bounded balls of positive integer radii, equipped with the metric coming from the trace. Each sort is then equipped with the restriction of the $*$-algebra structure; it is important to note that the proof that the algebra multiplication is uniformly continuous makes use of the tracial property of the state.

It is natural to try to extend the model-theoretic treatment of von Neumann algebras beyond the finite case. The next obvious class of von Neumann algebras to consider are the σ-finite ones. Since such algebras can be characterized as those which admit a faithful, normal state, one can try to mimic the case of finite von Neumann algebras by treating W*-probability spaces (M, φ) as metric structures in the same way as in the case of tracial W*-probability spaces. This time, one runs into the immediate issue that the algebra multiplication restricted to operator norm bounded balls may no longer be uniformly continuous with respect to the metric coming from the state, whence one must find a different approach. In [4], Dabrowski axiomatized the class of W*-probability spaces in a language that still used operator norm bounded balls as sorts, but which had to give up ordinary algebra multiplication in favor of a "smeared multiplication" obtained through an averaging procedure involving the modular automorphism group.

In this paper, we offer an alternative axiomatization of the class of W*-probability spaces motivated by the desire to have ordinary multiplication as part of the language. As a consequence, a compromise must be made in terms of the sorts. Since the issue in adapting the language for tracial W*-probability spaces to the non-tracial case stemmed from the fact that multiplication is not uniformly continuous with respect to the metric coming from the state, it becomes natural to change the sorts in such a way that multiplication does become uniformly continuous. Consequently, we take as our sorts the set of **totally K-bounded** elements of the W*-probability space (as K ranges over the natural numbers); here an element $x \in M$ is totally K-bounded if both left and right multiplication by both x and x^*, viewed as operators on the dense subspace $M\omega_\varphi$ of the GNS-Hilbert space H_φ, extend to bounded operators on H_φ of operator norm at most K. (Here, ω_φ is the vector in H_φ corresponding to the identity in M.) It is readily verified that all of the $*$-algebra operations, restricted to the set of totally K-bounded elements, are uniformly continuous with respect to the norm $\|\cdot\|_\varphi^\#$ arising from the state (see

§2 for the definition), which topologizes the strong* topology on operator norm bounded sets. Finally, since the set of totally bounded vectors can be shown to be strongly dense in the algebra, one can feel confident that one has not lost any information by restricting to such elements.

The proof that the category of models of our theory is equivalent to the category of W*-probability spaces is, in itself, mathematically interesting. Indeed, to establish this equivalence of categories, one needs to show, in particular, that if one begins with a model A of our theory, then forms the corresponding W*-probability space (M, φ) obtained as the strong closure of the direct limit of the sorts, and then "dissects" again to obtain a model A', then naturally A is isomorphic to A'. In the model-theoretic treatments of C*-algebras and of tracial von Neumann algebras, the analogous step is quickly dispensed with by standard functional calculus techniques. The aforementioned functional calculus arguments become problematic in the non-tracial setting, however, due to the necessity of handling left and right norms simultaneously. More to the point, it is a priori possible that after completing A, (M, φ) might have new totally bounded elements than those coming from A, whence A' would be strictly larger than A. In order to show that this does not in fact happen, in Section 4 we use ideas of Kadison [11] to obtain a novel characterization of the totally bounded elements of a W*-probability space, a result which we believe is of independent interest.

A crucial tool in the study of non-tracial W*-probability spaces is the **modular automorphism group** σ_t^{φ} of the state φ. In Dabrowski's approach, the modular automorphism group is simply part of the language. Since the model-theoretic ultraproduct construction associated to our language described above corresponds to the Ocneanu ultraproduct construction of W*-probability spaces (a fact established in Section 6) and the modular automorphism group is known to commute with the Ocneanu ultraproduct (as shown by Ando and Haagerup in [1]), general Beth Definability considerations imply that one may add symbols for the modular automorphism group to our language as to obtain a definitional expansion of our theory which captures W*-probability spaces equipped with their modular automorphism group. However, for reasons connected with recent applications of these ideas to computability concerns [2], it is important to have concrete, effectively given axioms for the modular automorphism group. We use ideas from the bounded operator approach to Tomita-Takesaki theory given by Rieffel and Van Daele in [14] (summarized in Section 3) to give an effective axiomatization of the modular automorphism group in a natural extension of our earlier language; this is undertaken in Section 7. (To be fair, part of our axiomatization from Section 5 also uses a bit of the Rieffel and Van Daele approach.)

In the final section of the paper, we mention some (non-)axiomatizability results for various classes of factors viewed as structures in our language. Most of these

results follow from results of Ando and Haagerup [1] viewed through a model-theoretic lens.

We assume familiarity with basic von Neumann algebra theory, although we review the facts we need about W*-probability spaces and GNS representations in the next section. We also assume familiarity with basic continuous model theory; the reader may consult [5] or [6] for introductions to continuous model theory as it pertains to operator algebras. Although not essential, familiarity with the axiomatization of tracial von Neumann algebras given in [6] (see also [8]) will help the reader appreciate the nuances encountered when moving to the σ-finite setting. For Tomita-Takesaki theory, [16] is a standard reference.

2. Preliminaries on W*-probability spaces

As mentioned in the introduction, the class of structures we wish to capture model-theoretically are the σ-finite von Neumann algebras, by which we mean those von Neumann algebras M for which any set $(p_i)_{i\in I}$ of pairwise orthogonal projections from M must contain only countably many nonzero elements. It is well-known that these algebras can be alternatively characterized as those which admit a faithful normal state; we will use this characterization to view a σ-finite von Neumann algebra as a metric structure.

However, it will behoove us to consider yet another characterization of this class. Fix a faithful, normal representation $\pi : M \to B(H)$ of M. We recall that $\omega \in H$ is called **cyclic** for π if $\pi(M)\omega$ is dense in H and **separating** for π if ω is cyclic for $\pi(M)'$, that is, if $\pi(a)\omega \neq 0$ for all $a \in M \setminus \{0\}$. π is then called a **standard representation** if it admits a cyclic and separating vector ω, in which case one says that (M, π, ω) is a **standard form** of M. It is convenient to abuse notation and speak simply of (M, ω) as a standard form of M, conflating the distinction between M and $\pi(M)$. One then has that M is σ-finite if and only if it admits a standard representation; moreover, any two such standard representations of M are unitarily conjugate.

If (M, ω) is a standard form of M, then the state $\varphi_\omega(a) := \langle a\omega, \omega\rangle$ is a faithful, normal state on M. Conversely, given a faithful, normal state φ on M, one obtains a standard form of M using the GNS construction, which we briefly recall. First, one introduces the inner product $\langle \cdot\,,\, \cdot \rangle_\varphi$ on M given by

$$\langle a\,,\, b\rangle_\varphi = \varphi(b^*a)$$

with corresponding norm

$$\|a\|_\varphi = \sqrt{\langle a\,,\, a\rangle_\varphi}.$$

The Hilbert space completion of $(M, \langle \cdot\,,\, \cdot \rangle_\varphi)$ is denoted by H_φ and elements of M, when viewed as elements of H_φ, are often decorated with a hat, so $a \in M$ would

be denoted $\hat{a}$. Given $a \in M$, one gets a bounded operator $\pi_\varphi(a) \in B(H_\varphi)$ defined on the dense subset $\widehat{M}$ of H_φ by setting $\pi_\varphi(a)(\hat{b}) := \widehat{ab}$. It is then easy to see that $\pi_\varphi : M \to B(H_\varphi)$ is a standard representation of M with cyclic and separating vector $\omega_\varphi := \hat{1}$. We note also that the topology on M induced by $\|\cdot\|_\varphi$ agrees with the strong topology on operator norm bounded sets and, moreover, that the associated metric is complete on such sets.

Until further notice, fix a standard form (M, ω) of M. In general, for $a \in M$, the right multiplication map $b\omega \mapsto ba\omega : M\omega \to M\omega$ does not extend to a bounded operator on H. Thus, we say that $a \in M$ is **right K-bounded** if, for all $b \in M$, we have $\|ba\| \leq K\|b\|$. Note that some sources in the literature would call such an element right $\sqrt{K}$-bounded. Note also that the notion of bounded element depends on the choice of standard form of M. If a is right K-bounded, then the above map extends to a bounded operator on H, which we denote by $\pi'(a)$; in this case, we set $\|a\|_{\text{right}} := \|\pi'(a)\|$ and note that $\|a\|_{\text{right}} \leq K$. In a slight abuse of notation, for $\xi \in H$ we will sometimes write $\|\pi'(\xi)\| \leq K$ if $M\omega \ni a\omega \mapsto a\xi$ extends to a K-bounded operator on H. In this way we have $\pi'(a) = \pi'(a\omega)$. We say that $a \in M$ is **right bounded** if it is right K-bounded for some K; we let M_{right} denote the set of right bounded elements of M.

Note that if $a \in M_{\text{rb}}$, then $\pi'(a) \in M'$. More generally, we have:

Lemma 2.1. *For $v \in H$, we have that $v \in M'\omega$ if and only if the map*

$$b\omega \mapsto bv : M\omega \to H$$

extends to a bounded operator on H. In particular, $a \in M$ is right-bounded if and only if $a\omega \in M'\omega$, in which case $\pi'(a)$ is the unique $z \in M'$ such that $a\omega = z\omega$.

Proof. First suppose that $v \in M'\omega$. Then $v = x\omega$ for some $x \in M'$. For $b \in M$, it follows that $bv = bx\omega = xb\omega$, whence the map $b\omega \mapsto bv$ is the restriction of the bounded operator x to $M\omega$.

Conversely, suppose that the map $b\omega \mapsto bv : M\omega \to H$ extends to the bounded operator x on H. Then $v = 1 \cdot v = x(1\omega) = x\omega$. It suffices to show that $x \in M'$. To see this, take $a, b \in M$ and note $(xa)(b\omega) = x(ab\omega) = abv = ax(b\omega)$; since $M\omega$ is dense in H, it follows that $ax = xa$. Since $a \in M$ is arbitrary, it follows that $x \in M'$.

The final statement follows immediately from the first statement and its proof. □

Continuing with the assumption that (M, ω) is a standard form of M, we say $x \in M$ is **totally K-bounded** if $\|x\| \leq K$ and both x and x^* are right K-bounded. We say that $x \in M$ is **totally bounded** if x is totally K-bounded for some K. We will denote the set of totally right bounded elements by M_{tb}.

Notation 2.2. For any $X \subseteq M$, we let $S_K(X)$ denote the set of totally K-bounded elements of X.

Now suppose that (M, φ) is a W*-probability space, by which we mean a von Neumann algebra equipped with a faithful, normal state φ. We always associate to such a space the standard form (M, ω_φ) stemming from the GNS construction as discussed above and any mention of (totally) bounded elements are relative to this standard form on M.

The following fact is well-known to experts and follows, for instance, from [16, Chapter VIII, Theorem 3.17].

Proposition 2.3. *Suppose that (M, φ) is a W*-probability space. Then M_{tb} is strongly dense in M.*

Since the previous proposition is crucial for our model-theoretic applications, we furnish a short, self-contained proof of this fact. Before proving Proposition 2.3, we need a lemma:

Lemma 2.4. *Let M be a von Neumann algebra and φ and ψ be two faithful, normal states on M. Then there is an increasing sequence p_K of projections strongly converging to 1 so that $\varphi(p_K x p_K) \leq K\psi(p_K x p_K)$ for all $x \in M_+$.*

Proof. Set $\theta_K := K\psi - \varphi$. Each θ_K is a hermitian, normal linear functional on M, whence by the Jordan decomposition [16, Chapter III, Theorem 4.2] there is a projection p_K such that

$$\theta_{K,+}(x) := \theta_K(p_K x p_K) \text{ and } \theta_{K,-}(x) := -\theta_K(p_K^\perp x p_K^\perp)$$

are positive and $\theta_K = \theta_{K,+} - \theta_{K,-}$. For $K \geq 1$, $p_K \neq 0$ and the sequence (p_K) is increasing in K. Set $p := \bigvee_K p_K$. Assume, towards a contradiction, that $p \neq 1$. Then we have that $\theta_K(p^\perp) = -\theta_{K,-}(p^\perp) \leq 0$ whence $\varphi(p^\perp) \geq K\psi(p^\perp)$ for all K, which is absurd since ψ is faithful. □

Proof of Proposition 2.3. Fix a unitary $u \in M$; it suffices to show that u is the strong limit of a sequence of right bounded elements of M. By the previous lemma, there is a sequence p_K of projections converging strongly to 1 so that, for all $x \in M_+$, we have $\varphi(u^* p_K x p_K u) \leq K\varphi(p_K x p_K)$. Since the sequence $p_K u$ converges strongly to u, it suffices to check that each $p_K u$ is right $\sqrt{K}$-bounded. Towards that end, fix $x \in M$ and observe that

$$\|x p_K u\|_\varphi^2 = \varphi(u^* p_K x^* x p_K u) \leq K\varphi(p_K x^* x p_K) \leq K\|x\|_\varphi^2,$$

whence $p_K u$ is $\sqrt{K}$-bounded.

It remains to check that $(p_K u)^* = u^* p_K$ is also right $\sqrt{K}$-bounded. To see this, fix $x \in M$ and observe that

$$\|xu^* p_K\|_\varphi^2 = \|xu^* p_K uu^*\|_\varphi^2 \leq \|xu^* p_K u\|_\varphi^2 \leq K\|xu^*\|_\varphi^2 \leq K\|x\|_\varphi^2.$$

This completes the proof. □

If (M, φ) is a W*-probability space, we also consider the norm $\|\cdot\|_\varphi^\#$ on M given by

$$\|a\|_\varphi^\# := \sqrt{\frac{\varphi(a^*a + aa^*)}{2}} = \sqrt{\frac{\|a\|_\varphi^2 + \|a^*\|_\varphi^2}{2}}.$$

Note that when φ is a trace, then $\|\cdot\|_\varphi = \|\cdot\|_\varphi^\#$. It is clear that $\|a^*\|_\varphi^\# = \|a\|_\varphi^\#$ and $\|a\|_\varphi^\# \leq \|a\|$ for all $a \in M$. Since the norm $\|\cdot\|_\varphi$ induces the strong topology on operator norm bounded subsets of M, it follows that the norm $\|\cdot\|_\varphi^\#$ induces the strong-* topology on operator norm bounded subsets of M and that such sets are strong-* closed.

Lemma 2.5. *For all $a \in M$, we have $|\varphi(a)| \leq \sqrt{2}\|a\|_\varphi^\#$.*

Proof. The statement follows immediately from Cauchy-Schwarz. □

Lemma 2.6. *Suppose that (M, φ) is a W*-probability space. Then for any K, the strong and strong* topologies agree on totally K-bounded sets.*

Proof. We show that on totally K-bounded subsets, the norms $\|\cdot\|_\varphi$ and $\|\cdot\|_\varphi^\#$ are equivalent. By definition, $\|a\|_\varphi^\# \geq \frac{1}{\sqrt{2}}\|a\|_\varphi$ for all $a \in M$. Conversely, if a is totally K-bounded, then

$$\|a\|_\varphi^\# \leq \sqrt{\frac{\|a^*\|\|a\|_\varphi + \|a^*\|_{\text{right}}\|a\|_\varphi}{2}} \leq \sqrt{K}\|a\|_\varphi.$$

□

As stated in the introduction, when treating a W*-probability space as a many-sorted structure, our intended sorts are the $S_K(M)$'s. We will equip these sorts with the metric stemming from the norm $\|\cdot\|_\varphi^\#$. Consequently, we will need to know that our sorts are complete with respect to this metric, a fact we now prove.

First, we need to introduce some important operators associated to any standard form. Suppose that (M, ω) is a standard form of M. One then defines the unbounded operators $S_0 : M\omega \to H$ and $F_0 : M'\omega \to H$ by $S_0(a\omega) := a^*\omega$ and $F_0(b\omega) := b^*\omega$ for $a \in M$ and $b \in M'$. An important fact about these unbounded operators are that they are closable [16, II, Lemma 1.5]; we set $S := \overline{S_0}$ and $F := \overline{F_0}$. One also has that $S^* = F$ and $S^2 = F^2$ equals the identity on its

(dense) domain. Given a W*-probability space, we will denote by S_φ and F_φ the associated operators on the GNS construction.

Lemma 2.7. *For each K, the set $S_K(M)$ is $\|\cdot\|_\varphi^\#$-complete.*

Proof. Suppose that $(a_n)_{n\in\mathbb{N}}$ is a $\|\cdot\|_\varphi^\#$-Cauchy sequence from $S_K(M)$. Since operator norm bounded balls are $\|\cdot\|_\varphi$-complete, there is $a \in M$ such that a_n strongly converges to a. We claim that a is also right K-bounded. To see this, take $b \in M$ and note that ba_n converges strongly to ba and hence $\|ba_n\|_\varphi$ converges to $\|ba\|_\varphi$. Since $\|a_n\|_{\text{right}} \leq K$, we have that $\|ba\|_\varphi = \lim_{n\to\infty} \|ba_n\|_\varphi \leq K\|b\|_\varphi$, whence $\|a\|_{\text{right}} \leq K$.

Arguing as in the previous paragraph, there is $a' \in M$ such that a_n^* strongly converges to a'; moreover, $\|a'\|, \|a'\|_{\text{right}} \leq K$.

It remains to show that $a^* = a'$. Indeed, we have that $(a_n - a)\omega \to 0$, whence by virtue of S_φ being closed, we have that $(a_n^* - a^*)\omega = S_\varphi(a_n - a)\omega \to 0$. It follows that a_n^* strongly converges to a^* and thus $a' = a^*$. □

We close this section with some words about embeddings between probability spaces. An **embedding** between W*-probability spaces (M, φ) and (N, ψ) is a unital, normal, state-preserving *-homomorphism $f : M \to N$ for which there is a normal, state-preserving conditional expectation map $E : N \to M$.

The reason for this choice of embeddings comes from modular theory (see the next lemma). Notice that in the tracial setting, every trace-preserving *-homomorphism automatically induces a conditional expectation onto its image. This corresponds to the fact that, in this setting, the modular automorphism group is trivial.

In order to show that we axiomatized the category of W*-probability spaces, we will need the following alternate characterization of embeddings between W*-probability spaces. Note that this characterization refers to the modular automorphism group of a W*-probability space, defined in detail in the next section. Throughout this paper, we let M_{sa} denote the set of self-adjoint elements of M.

Lemma 2.8. *Suppose that (M, φ) and (N, ψ) are W*-probability spaces and $f : M \to N$ is a state-preserving *-homomorphism. Then the following are equivalent:*

(1) f is an embedding of W-probability spaces.*
(2) f preserves distances to the sets of self-adjoint elements, that is, for all $a \in M$, we have $d_\varphi(a, M_{\text{sa}}) = d_\psi(f(a), N_{\text{sa}})$.
(3) f commutes with the modular automorphism group, that is, for every $t \in \mathbb{R}$ and $a \in M$, we have $f(\sigma_t^\varphi(a)) = \sigma_t^\psi(f(a))$.

Proof. The direction (1) implies (2) is clear. The direction (2) implies (3) follows from Rieffel and Van Daele's bounded operator approach to Tomita-Takesaki theory described in the next section. The direction (3) implies (1) is a theorem of Takesaki [16, Chapter IV, Theorem 4.2]. □

In connection with the previous lemma, we note that if $a \in S_K(M)$, then $d_\varphi(a, M_{\text{sa}}) = d_\varphi(a, b)$ for some $b \in S_{3K}(M)$; this fact will be proven in the next section.

3. Tomita-Takesaki theory according to Rieffel and Van Daele

In this section, we present the basic facts from the bounded operator approach to Tomita-Takesaki due to Rieffel and Van Daele [14] used throughout this paper.

Fix a standard form (M, ω) of M. We set $\mathcal{K} := \overline{M_{\text{sa}}\omega}$, a closed, *real* subspace of H. Viewing H as a real Hilbert space by taking the real part of the original inner product, we let $P : H \to H$ denote the orthogonal projection onto $\mathcal{K}$. We furthermore set $Q : H \to H$ to be the orthogonal projection onto the closed real subspace $i\mathcal{K}$ of H and set $R := P + Q$, a positive operator. Moreover, one considers the polar decomposition of $\Theta := P - Q = JT$ of $P - Q$, where J is a partial isometry and T is a positive operator. We list here the most important facts about these operators and spaces, whose proofs can be found in [13, 14]:

Facts 3.1.

(1) R and T are complex-linear operators.
(2) J is an anti-linear, self-adjoint, orthogonal operator (whence $J^2 = 1$) satisfying $J\omega = \omega$.
(3) R and $2 - R$ are injective.
(4) For all $v \in H$, $iP(v) = Q(iv)$.
(5) $\overline{M'_{\text{sa}}\omega} = i\mathcal{K}^\perp$.
(6) If $x \in M$, then $\Theta x\omega = (2 - R)x^*\omega$.
(7) If $x' \in M'$, then $\Theta x'\omega = Rx'^*\omega$

Since the spectra of R and $2-R$ are both contained in $(0, 2)$ and the function $x \mapsto x^{it}$ is bounded and continuous on this interval, one may use functional calculus to define the operators R^{it} and $(2 - R)^{it}$ for all $t \in \mathbb{R}$, yielding two commuting strongly continuous one-parameter groups of unitaries of H. One then defines a new strongly continuous one-parameter group of unitaries $\Delta^{it} := (2 - R)^{it}R^{-it}$.

The notation Δ^{it} may seem strange since one has not defined the operator Δ itself; in the usual (unbounded) approach to Tomita-Takesaki theory, one indeed defines the unbounded operator Δ first and then uses (unbounded) functional calculus to define the unitary operators Δ^{it}. One can describe Δ using the approach discussed

above, namely $\Delta := R^{-1}(2-R)$; the presence of the unbounded operator R^{-1} renders Δ unbounded.

It will behoove us to recall the "standard" approach to defining Δ. Indeed, recalling the closed, unbounded operators S and F introduced in the previous section, the usual definition of Δ is as the positive operator in the polar decomposition of S, that is $S = J\Delta^{1/2}$, where J is as defined above. Note then that $\Delta = S^*S = FS$.

We next observe how this picture looks when we switch focus from M to M'. Let us temporarily decorate the corresponding operators with a $'$, such as S', F', etc... It is clear that $S' = F$, $F' = S$, $\Delta' = F'S' = SF = \Delta^{-1}$. By item (5) of Facts 3.1, we see that $P' = 1 - Q$. Analogously, one has that $Q' = 1 - P$. Consequently, $\Theta' = P' - Q' = (1-Q) - (1-P) = P - Q = \Theta$. Note also that $R' = P' + Q' = (1-Q) + (1-P) = 2 - R$.

The following is [14, Corollary 4.4].

Lemma 3.2. *For every $x' \in M'$, there exists a unique $x \in M$ such that*

$$\Theta x'\omega = x\omega \quad \text{and} \quad \Theta(x')^*\omega = x^*\omega.$$

By interchanging M and M', we also have that, for every $y \in M$, there exists a unique $y' \in M'$ so that

$$\Theta y\omega = y'\omega \quad \text{and} \quad \Theta y^*\omega = (y')^*\omega.$$

For our purposes, we will need a sharpening of the previous lemma. In what follows, for $v \in M\omega$, we abuse notation by setting $\|\pi(v)\| := \|a\|$, where $a \in M$ is the unique element of M for which $v = a\omega$.

Lemma 3.3. *If $x \in M'_K$ is such that $\|x\| \leq K$, then $P(x\omega), Q(x\omega) \in M\omega$ and $\|\pi(Px\omega)\|, \|\pi(Qx\omega)\| \leq 2K$.*

Proof. First suppose that $x' \in M'$ is such that $0 \leq x' \leq 1$. In the proof of [14, Lemma 4.3], the authors find $x \in M_{\text{sa}}$ with $\|x\| \leq 1$ such that $P(x'\omega) = x\omega$. Consequently, for general positive $x' \in M'$, there is $x \in M_{\text{sa}}$ such that $\|x\| \leq \|x'\|$ satisfying $P(x'\omega) = x\omega$.

If $x' \in M'_{\text{sa}}$ is not necessarily positive, then we may write $x' = x'_1 - x'_2$ with $x'_1, x'_2 \in M'_{\text{sa}}$ positive and $\|x'_i\| \leq \|x'\|$; it follows that

$$P(x'\omega) = P(x'_1\omega) - P(x'_2\omega) = x_1\omega - x_2\omega = (x_1 - x_2)\omega,$$

where $x_1, x_2 \in M_{\text{sa}}$ are such that $\|x_1 - x_2\| \leq \|x_1\| + \|x_2\| \leq \|x'_1\| + \|x'_2\| \leq 2\|x'\|$. The claim follows.

Noting that $P = -iQi$, the same proof gives the statement for Q. □

An immediate corollary of Lemma 3.3 is the following:

Corollary 3.4. *If $a \in M_{\mathrm{tb}}$, then $P(a\omega) = b\omega$ for a unique $b \in M$ with $\|b\| \leq 2\|a\|_{\mathrm{right}}$.*

We now prove an upper bound for the right norm of b as in the previous corollary:

Lemma 3.5. *Fix $a \in M_{\mathrm{tb}}$ and write $P(a\omega) = b\omega$ for $b \in M$. Then $b \in M_{\mathrm{tb}}$ and $\|b\|_{\mathrm{right}} \leq \|a\|_{\mathrm{right}} + 2\|a\|$.*

Proof. First, recalling that ω is also a cyclic and separating vector for $M' \subseteq \mathcal{B}(H_\varphi)$, we set $\mathcal{K}' := \overline{M'_{\mathrm{sa}}\omega}$ and let P' denote the projection onto $\mathcal{K}'$. By Fact 3.1(5), we have that $\mathcal{K}' = i\mathcal{K}^\perp$, whence $P' = 1 - Q$. By Lemma 3.3 applied to P' and using the fact that $a \in M = (M')'$, we have that $P'(ia\omega) = c\omega$ with $c \in M'$ satisfying $\|c\| \leq 2\|a\|$. On the other hand, $P'(ia\omega) = (1 - Q)(ia\omega) = ia\omega - iP(a\omega)$, so $P(a\omega) = a\omega - ic\omega$. Recalling that $P(a\omega) = b\omega$, we have, for any $d \in M$, that

$$\|db\|_\varphi = \|da\omega - d(ic\omega)\|_\varphi \leq \|a\|_{\mathrm{right}}\|d\|_\varphi + \|c\|\|d\|_\varphi \leq (\|a\|_{\mathrm{right}} + 2\|a\|)\|d\|_\varphi.$$

It follows that $b \in M_{\mathrm{tb}}$ and $\|b\|_{\mathrm{right}} \leq \|a\|_{\mathrm{right}} + 2\|a\|$. □

Corollary 3.6. *If $a \in S_K(M)$, then $P(a\omega) = b\omega$ for a unique $b \in S_{3K}(M)$.*

Proof. By Corollary 3.4 and Lemma 3.5, we know that $\|b\| \leq K$ and $\|b\|_{\mathrm{right}} \leq 3K$. On the other hand, by the construction in Proposition 3.3, we have that $P(a^*\omega) = b^*\omega$; since a^* is also φ-right K-bounded, Lemma 3.5 implies that $\|b^*\|_{\mathrm{right}} \leq 3K$. It follows that $b \in S_{3K}(M)$. □

The main theorem of Tomita-Takesaki theory is the following:

Theorem 3.7 (Theorem VI.1.19 in [16])**.** *$JMJ = M'$ and $\Delta^{it}M\Delta^{-it} = M$ for all $t \in \mathbb{R}$.*

By the previous fact, one can define a strongly continuous one-parameter family σ_t of automorphisms of M by defining $\sigma_t(x) := \Delta^{it}x\Delta^{-it}$. This family is called the **modular automorphism group** of the standard form (M, ω).

We now specialize to the case that (M, φ) is a W*-probability space with standard form stemming from the GNS representation. As mentioned in Section 2, sometimes we identify M with its image $\pi_\varphi(M)$. Consequently, M' here really refers to $\pi_\varphi(M)'$. We occasionally might decorate the various operators just introduced with φ's, such as P_φ, to indicate that we are working in this context. In particular, we write σ_t^φ for the aforementioned one-parameter family of automorphisms of M and refer to this family as the **modular automorphism group of φ.**

In Section 7, we explain how the Rieffel-Van Daele approach to Tomita-Takesaki theory can be used to expand the language given in Section 5 by symbols for the modular automorphism group of φ and to extend the theory presented in Section

5 to an effective theory in the expanded language axiomatizing the modular automorphism group. In order to do so, we will need the following two facts, the first of which is a special case of [14, Theorem 5.15]:

Fact 3.8. *For each $a \in M$, there is unique $b \in M$ such that $\Delta_\varphi^{it}(a\omega) = b\omega$. Moreover, we have $\pi_\varphi(b) = \Delta_\varphi^{it}\pi_\varphi(a)\Delta_\varphi^{-it}$. Consequently, $\|a\| = \|b\|$.*

Note that by taking adjoints in the previous fact, we see also that $\Delta^{it}(a^*\omega) = b^*\omega$. The second fact is a special case of [14, Theorem 5.14]:

Fact 3.9. *For all $a, b \in M$ and $t \in \mathbb{R}$, we have*

$$(\Delta_\varphi^{it} J_\varphi \pi_\varphi(a) J_\varphi \Delta_\varphi^{-it})(b\omega) = \pi_\varphi(b)(\Delta_\varphi^{it}(J_\varphi a\omega)).$$

We now return to the general case of a standard form (M, φ). We need one further fact about the modular operator, to be used in the next section. See [16, VI, Lemma 1.17].

Fact 3.10 (Bridging Lemma/Takesaki's Resolvent Lemma)**.** *For every $x \in M$ and $\lambda \in \mathbb{C} \setminus \mathbb{R}_+$, there exists a unique $y' \in M'$ such that $(\Delta^{-1} - \lambda I)^{-1} x\omega = y'\omega$. Furthermore, one has*

$$\|y'\| \leq \frac{\|x\|}{\sqrt{2|\lambda| - 2\operatorname{Re}(\lambda)}}.$$

Similarly, for every $x' \in M'$ and $\lambda \in \mathbb{C} \setminus \mathbb{R}_+$, there exists a unique $y \in M$ such that $(\Delta - \lambda)^{-1} x'\omega = y\omega$. Futhermore, one has

$$\|y\| \leq \frac{\|x'\|}{\sqrt{2|\lambda| - 2\operatorname{Re}(\lambda)}}.$$

We will need a "totally bounded" version of the previous fact:

Corollary 3.11. *If $x \in M$ (resp. $x' \in M'$) is right bounded, then $y' \in M'$ (resp. $y \in M$) as in Fact 3.10 is totally bounded with*

$$\|y'\|_{\text{right}}, \|(y')^*\|_{\text{right}} \leq \frac{\|x\|_{\text{right}}}{\sqrt{2|\lambda| - 2\operatorname{Re}(\lambda)}}$$

(and similarly for the bounds on $\|y\|_{\text{right}}, \|y^*\|_{\text{right}}$).

Proof. By symmetry, it suffices to prove the statement for $x \in M$. Since $x\omega = \pi'(x)\omega$ with $\pi'(x) \in M'$, by Fact 3.10, there is $a \in M$ such that $(\Delta^{-1} - \lambda I)\pi'(x)\omega = a\omega$ with $\|a\| \leq \frac{\|\pi'(x)\|}{\sqrt{2|\lambda| - 2\operatorname{Re}(\lambda)}}$. Since $y'\omega = a\omega$, we have that $\|y'\|_{\text{right}} = \|a\|$.

Now

$$\pi'(x)\omega = x\omega = (\Delta^{-1} - \lambda I)y'\omega = SFy'\omega - \lambda y'\omega,$$

so

$$\pi'(x)^*\omega = F\pi'(x)\omega = FSFy'\omega - \lambda Fy'\omega = (\Delta - \lambda I)Fy'\omega,$$

and thus

$$(\Delta - \lambda I)^{-1}(\pi'(x)^*\omega) = Fy'\omega.$$

By Fact 3.10, there is $b \in M$ such that $(\Delta - \lambda I)^{-1}(\pi'(x)^*\omega) = b\omega$ and

$$\|b\| \leq \frac{\|\pi'(x)^*\|}{\sqrt{2|\lambda| - 2\operatorname{Re}(\lambda)}} = \frac{\|\pi'(x)\|}{\sqrt{2|\lambda| - 2\operatorname{Re}(\lambda)}}.$$

Since $(y')^*\omega = b\omega$, we have the desired bound on $\|(y')^*\|_{\text{right}}$. □

4. Characterization of M_{tb}

In this section, we prove a characterization of the set of totally bounded elements of a W*-probability space, to be used in our axiomatization of W*-probability spaces in the next section. One of the items in the characterization involves the operator $h_a(\log(\Delta))$, where $h_a(t)$ is the following function, defined for all $a \in \mathbb{R}$:

$$h_a(t) := 1/(\cosh(t-a)) = \frac{2}{e^{t-a} + e^{a-t}}.$$

Throughout this section, we fix a W*-probability space (M, φ) with associated standard form (M, ω).

Proposition 4.1. *For every $a \in \mathbb{R}$ and $x \in M_{\text{tb}}$, there exists $y \in M_{\text{tb}}$ satisfying $h_a(\log(\Delta))x\omega = y\omega$. Furthermore, $\|y\| \leq \|x\|$ and $\|y\|_{\text{right}} \leq \|x\|_{\text{right}}$.*

Proof. Note that

$$h_a(\log(\Delta)) = 2(e^{-a}\Delta + e^a\Delta^{-1})^{-1} = 2i(\Delta + ie^aI)^{-1}(\Delta^{-1} + ie^{-a}I)^{-1}.$$

The proposition then follows from two applications of Corollary 3.11. □

The following easy calculation will prove useful in the next section:

Lemma 4.2. *For $x, y \in M$, we have $h_a(\log(\Delta))x\omega = y\omega$ if and only if*

$$2R(2-R)x\omega = (e^{-a}(2-R)^2 + e^aR^2)y\omega.$$

The following is the main result of this section:

Theorem 4.3. *Suppose that M_0 is a strongly dense $*$-subalgebra of M such that:*

(1) $M_0 \subseteq M_{\text{tb}}$.
(2) $S_1(M_0)$ is $\|\cdot\|_\varphi^\#$-complete.
(3) $h_a(\log\Delta)(M_0) \subseteq M_0$ for all $a \in \mathbb{N}$.

Then $M_0 = M_{\mathrm{tb}}$.

Remark 4.4. At this point, the stipulation of closure of M_0 under $h_a(\log(\Delta)$ may appear to be lacking a clear motivation. However, given a W*-probability space (M, φ), the restriction of the spectral subspaces of $\log(\Delta)$ to $M\omega$ are dense and can be seen to define the spectral subspaces $M(\sigma^\varphi, [a, b])$ of (M, φ) by

$$M(\sigma^\varphi, [a, b]) = \{x \in M : \chi_{[a,b]}(\log(\Delta))x\omega = x\omega\}.$$

The spectral subspaces have been a crucial tool in the study of the structure of W*-probability spaces since the seminal work of Connes [3]. In this light, it is natural to want that M_0 densely intersects the spectral subspaces, and we have seen that "smoothed out" approximations $h_a(\log(\Delta))$ to $\log(\Delta)$ can be expressed in the bounded framework of Rieffel and Van Daele on which our axiomatic treatment of W*-probability spaces will rely. Subsequently, we will crucially use the analysis of the spectral subspaces of an ultraproduct of W*-probability spaces developed by Ando and Haagerup [1] to show that that Ocneanu ultraproduct agrees with the ultraproduct given by this axiomatization.

We prove Theorem 4.3 in a series of steps. First, for each $a \in \mathbb{R}$, we introduce two more functions:

$$f_a(t) = e^{-|t-a|}$$

and

$$g_a(t) = e^{-|t|} - \frac{e^{-|t-a|} + e^{-|t+a|}}{e^a + e^{-a}}.$$

The following is [11, Lemma 4.11]:

Fact 4.5. *For all* $a \in \mathbb{R}$ *and* $x \in M$*, there exists* $y \in M$ *such that* $f_a(\log(\Delta))x\omega = y\omega$*. Furthermore,* $\|y\| \leq \|x\|$.

Temporarily, let us call a subalgebra M_0 satisfying the hypotheses of Theorem 4.3 **good**.

Proposition 4.6. *If* M_0 *is a good subalgebra of* M*, then for all* $a \in \mathbb{N}$ *and* $x \in M_0$*, there are* $y, z \in M_0$ *such that* $f_a(\log(\Delta))(x\omega) = y\omega$ *and* $g_a(\log(\Delta))(x\omega) = z\omega$.

Proof. Write

$$e^{-|t|} = \sum_{n=0}^{\infty} a_n(\cosh(t))^{-(2n-1)}$$

where each a_n is a positive real and $\sum_{n=0}^{\infty} a_n = 1$ as in the proof of lemma 4.11 in [11]. Plugging in $\log(\Delta)$ gives

$$f_a(\log(\Delta)) = \sum_{n=0}^{\infty} a_n (h_a(\log(\Delta)))^{2n-1},$$

where the sum is norm-convergent. Thus, for every $x \in M_0$ which is totally K-bounded, closure under $h_a(\log(\Delta))$ implies the existence of $y_n \in M_0$ such that $y_n\omega = (h_a(\log(\Delta)))^{2n-1}x$. Moreover, by repeated application of Proposition 4.1, each y_n is totally K-bounded. Since each a_n is a positive real and $\sum_{n=0}^{\infty} a_n = 1$, setting $b_k := \sum_{n=0}^{k} a_n y_n$, we see that each b_k is totally K-bounded and $b_k\omega$ converges to $f_a(\log(\Delta))x\omega$. Reasoning as in Corollary 3.11 shows that $(b_k)_{k\in\mathbb{N}}$ is in fact $\|\cdot\|_\varphi^\#$-Cauchy and thus b_k strong-* converges to some $y \in M_0$ by completeness of $S_K(M_0)$. We then have that $f_0(\log(\Delta))(x\omega) = y\omega$. The existence of $z \in M_0$ follows from closure under $f_a(\log(\Delta))$ and $f_0(\log(\Delta))$ by considering appropriate linear combinations. □

From now on, j denotes an element of $\mathbb{R}^{>1}$. We let $E(j^{-1}, j)$ denote the spectral projection of Δ (and Δ^{-1}) corresponding to the interval (j^{-1}, j) and set $E_j := E(j^{-1}, j)(H_\varphi)$.

The following is [11, Lemma 4.12]:

Fact 4.7. *Suppose that $x \in M$ and $x\omega \in E_j$. Then:*

1. *For all $n \in \mathbb{Z}$, there is $x_n \in M$ such that $\Delta^n(x\omega) = x_n\omega$ and $\|x_n\| \leq j^{|n|}\|x\|$.*
2. *x is right bounded and $\|x\|_{\text{right}} \leq j^{1/2}\|x\|$.*

The statement obtained by switching M and M' is also valid.

Corollary 4.8. *Suppose that $x \in M$ and $x\omega \in E_j$. Then x is totally $j^{3/2}\|x\|$-bounded.*

Proof. By Fact 4.7, $\|x\|_{\text{right}} \leq j^{\frac{1}{2}}\|x\|$. By Fact 4.7 again, we may write $\Delta(x\omega) = x_1\omega$ with $x_1 \in M$ and $\|x_1\| \leq j\|x\|$. Since E_j is invariant under Δ, we may apply the previous fact again to conclude that $\|x_1\|_{\text{right}} \leq j^{1/2}\|x_1\| \leq j^{3/2}\|x\|$. Note that $\Delta x\omega \in M'\omega$ by the previous fact and thus $Sx\omega = F\Delta x\omega$. Since $\pi'(Fy\omega) = \pi'(y)^*$ for all $y \in M'$, it follows that

$$\|x^*\|_{\text{right}} = \|\pi'(Sx\omega)\| = \|\pi'(Sx\omega)^*\| = \|\pi'(FSx\omega)\| = \|\pi'(\Delta x\omega)\| \leq j^{\frac{3}{2}}\|x\|.$$

This proves the corollary. □

Our next main goal is the following, which by Corollary 4.8, is a special case of Theorem 4.3:

Theorem 4.9. *Suppose that M_0 is a good subalgebra. If $x \in M$ is such that $x\omega \in E_j$, then $x \in M_0$.*

We begin with some lemmas. Until further notice, M_0 denotes a good subalgebra of M.

Lemma 4.10. *For all $x \in M$ and $a \in \mathbb{N}$, we have that $g_a(\log(\Delta))x \in M_0$ and $\|g_a(\log \Delta)(x)\|_{\text{right}} \leq 3e^{3a/2}\|x\|$.*

Proof. Without loss of generality, we may assume that $\|x\| = 1$. By Kaplansky density, we can find a sequence x_n of contractions in M_0 strongly converging to x. Then $y_n := g_a(\log(\Delta))x_n$ strongly converges to $g_a(\log(\Delta))x$. Moreover, $\|y_n\| \leq 3$ for all n. Since $y_n \in E_j$, where $j = e^a$, it follows that y_n is totally $3j^{3/2}$-bounded. The claim now follows by completeness of totally bounded subsets of M_0 and the fact that the strong and strong-* topologies are the same on totally bounded subsets. □

Lemma 4.11. *If $x \in M$ is such that $x\omega \in E_j$, then $\Delta^n g_j(\log(\Delta))x\omega \in M_0\omega$ for all integers n.*

Proof. By assumption and Fact 4.7, $\Delta^n x\omega \in M\omega$. Lemma 4.10 thus yields that $\Delta^n g_j(\log(\Delta))(x\omega) = g_j(\log(\Delta))(\Delta^n x\omega) \in M_0\omega$. □

For a fixed j, we split E_j into pieces E_-, E_c and E_+ corresponding to the intervals $(\frac{1}{j}, 1)$, $\{1\}$ and $(1, j)$ respectively. Define $k_+(t) := g_a(2t - a)$ and $k_- := g_a(2t + a)$, where $a := \log(j)$.

Lemma 4.12. *$E_+(H) \cap M\omega$ is dense in $E_+(H)$ and $E_-(H) \cap M\omega$ is dense in $E_-(H)$.*

Proof. It is clear from construction that $k_+(\log(\Delta))$ maps $M\omega$ to $M\omega$. Note that $k_+(t)$ is strictly positive on $(0, a)$ and 0 everywhere else. Thus $k_+(\log(\Delta))x$ is contained in the spectral subspace of Δ corresponding to $(1, j)$ for all x. Since k_+ does not vanish on $(0, a)$, we have $k_+(\log \Delta)(H)$ is dense in $E_+(H)$. Moreover, since $\{E_+(x\omega) : x \in M\}$ is dense in $E_+(H)$, we have that $\{k_+(\log(\Delta))(E_+(x\omega)) : x \in M\}$ is dense in $E_+(H)$. The proof for $E_-(H)$ is analogous. □

The following is a straightforward calculation:

Lemma 4.13. *If $v \in E_c(H)$, then*

$$g_j(\log \Delta)(v) = \frac{e^j - e^{-j}}{e^j + e^{-j}}v.$$

In connection with the next lemma, we note that $g_j(\log \Delta)$ is bounded and invertible (with bounded inverse) on E_j.

Lemma 4.14. $(g_j(\log\Delta)|E_-)^{-1}(M\omega\cap E_-), (g_j(\log\Delta)|E_+)^{-1}(M\omega\cap E_+)\subseteq M\omega$.

Proof. Suppose first that $x\omega\in E_-(H)$. Then

$$\begin{aligned} g_j(\log(\Delta))(x\omega) &= \left[\Delta - \frac{e^{-j}\Delta + e^{-j}\Delta^{-1}}{e^j+e^{-j}}\right](x\omega) \\ &= \frac{\Delta(e^j+e^{-j}) - (e^{-j}\Delta + e^{-j}\Delta^{-1})}{e^j+e^{-j}}(x\omega) \\ &= \frac{\Delta e^j - e^{-j}\Delta^{-1}}{e^j+e^{-j}}(x\omega) \end{aligned}$$

It thus suffices to show that $(e^j\Delta - e^{-j}\Delta^{-1})^{-1}(x\omega)\in M\omega$. Since

$$(e^j\Delta - e^{-j}\Delta^{-1})^{-1} = \Delta(e^j\Delta^2 - e^{-j})^{-1}$$

and $M\omega\cap E_j$ is closed under Δ, it suffices to show that $(e^{2j}\Delta^2-1)^{-1}(x\omega)\in M\omega$; however, this follows by writing $(e^{2j}\Delta^2-1)^{-1}$ as a geometric series and using that $x\omega$ is in the $(\frac{1}{j},1)$ spectral subspace of Δ.

Next suppose that $x\omega\in E_+(H)$. Then

$$\begin{aligned} g_j(\log(\Delta))(x\omega) &= \left[\Delta^{-1} - \frac{e^{-j}\Delta + e^{-j}\Delta^{-1}}{e^j+e^{-j}}\right](x\omega) \\ &= \frac{\Delta^{-1}(e^j+e^{-j}) - (e^{-j}\Delta + e^{-j}\Delta^{-1})}{e^j+e^{-j}}(x\omega) \\ &= \frac{\Delta^{-1}e^j - e^{-j}\Delta}{e^j+e^{-j}}(x\omega). \end{aligned}$$

It thus suffices to show that $(e^{-j}\Delta - e^j\Delta^{-1})^{-1}(x\omega)\in M\omega$. This time, write $(e^{-j}\Delta - e^j\Delta^{-1})^{-1} = \Delta^{-1}(e^{-j}-e^j\Delta^{-2})^{-1}$ and argue as in the previous case. □

Proof of Theorem 4.9. Without loss of generality, we may assume that $j\in\mathbb{N}$ and $\|x\|\leq 1$. Write $x\omega = v_- + v_c + v_+$, where $v_-\in E_-(H)$, $v_c\in E_c(H)$, and $v_+\in E_+(H)$. By Lemma 4.12, we may write $v_- = \lim a_n\omega$ with each $a_n\in M$ and $a_n\omega\in E_-(H)$ and $v_+ = \lim b_n\omega$ with each $b_n\in M$ and $b_n\omega\in E_-(H)$. Moreover, we may assume that $\|a_n\|,\|b_n\|\leq 1$ for each n. By Lemma 4.14, for each n, we may find $\widehat{a_n},\widehat{b_n}\in M$ such that $g_j(\log\Delta)^{-1}(a_n\omega) = \widehat{a_n}\omega$, $g_j(\log\Delta)^{-1}(b_n\omega) = \widehat{b_n}\omega$, and $\widehat{a_n}\omega,\widehat{b_n}\omega\in E_j$. By Corollary 4.8, we have that $\widehat{a_n}$ and $\widehat{b_n}$ are totally $j^{3/2}M$-bounded, where $M := \|(g_j(\log\Delta)|E_j)^{-1}\|$. Consequently, we have

$$x\omega = g_j(\log\Delta)(\lim_n \widehat{a_n}\omega) + v_c + g_j(\log\Delta)(\lim_n \widehat{b_n}\omega).$$

By Lemma 4.10 $g_j(\log\Delta)(\widehat{a_n}\omega) := a_n^\dagger\omega$ and $g_j(\log\Delta)(\widehat{b_n}\omega) = b_n^\dagger\omega$ for some $a_n^\dagger, b_n^\dagger\in M_0$; moreover, there is $K>0$ such that each $a_n^\dagger$ and $b_n^\dagger$ are totally K-bounded. Therefore, by completeness of $S_K(M)$, the first and third terms of the display

belong to $M_0\omega$. By Lemmas 4.10 and 4.13, the middle term also belongs to $M_0\omega$, whence the corollary is proven. □

In light of Theorem 4.9, in order to finish the proof of Theorem 4.3, one must approximate arbitrary totally bounded elements by ones that belong to compact spectral subspaces. We use the method of Bochner integrals found, for instance, in [16, II, Lemma VI.2.4].

For $v \in H$ and $r > 0$, we set

$$v_r = \sqrt{\frac{r}{\pi}} \int_{\mathbb{R}} e^{-rt^2} \Delta^{it} v \, \mathrm{dt} \, .$$

For all $z \in \mathbb{C}$, v_r belongs to the domain of Δ^{iz} and we have $\Delta^{iz} v_r = v_r(z)$, where $v_r(z) \in H$ is defined by

$$v_r(z) = \sqrt{\frac{r}{\pi}} \int_{\mathbb{R}} e^{-r(t-z)^2} \Delta^{it} v \, \mathrm{dt}$$

defines $\Delta^{iz} v_r$. Moreover, the map $z \mapsto v_r(z) : \mathbb{C} \to H$ is an entire function in the variable z. Consequently, for all $r > 0$, we have $v_r \in E_j$ for some $j > 1$ (see [16, Proof of Theorem VI.2.2.]).

The proof of the following is in [10, Lemma 1.3] with details drawing from the proof of [17, Lemma 10.1].

Fact 4.15. *Suppose that $x \in M$ and set $v := x\omega$. Then for each $r > 0$, there is $x_r \in M$ such that $v_r = x_r\omega$. Moreover, $\|x_r\| \leq \|x\|$. Furthermore, we have v_r strong* converges to v as $r \to 0$.*

Proof of Theorem 4.3. Suppose that $x \in M_{\mathrm{tb}}$ is a totally K-bounded element and set $v = x\omega$. By Theorem 4.9, each $v_r \in M_0$. Moreover, since all of the expressions above are symmetric under interchanging M and M', we have that each x_r is totally K-bounded. Since $S_K(M_0)$ is $\|\cdot\|_\varphi^\#$-complete, we have that $x \in M_0$. □

5. Axiomatizating W*-probability spaces

We are now ready to introduce our language $\mathcal{L}_{W^*}$ for W*-probability spaces:

(1) For each $n \in \mathbb{N}$, there is a sort S_n with bound $2n$, whose intended interpretation is the set $S_n(M)$. We let d_n denote the metric symbol on S_n, whose intended interpretation is the metric induced by $\|\cdot\|_\varphi^\#$.
(2) For each $n \in \mathbb{N}$, there are binary function symbols $+_n$ and $-_n$ with domain S_n^2 and range S_{2n} and whose modulus of uniform continuity is $\delta(\epsilon) = \epsilon$. The intended interpretation of these symbols are addition and subtraction in the algebra restricted to the sort S_n.

(3) For each $n \in \mathbb{N}$, there is a binary function symbol $\cdot_n$ with domain S_n^2 and range S_{n^2} and whose modulus of uniform continuity is $\delta(\epsilon) = \frac{\epsilon}{n}$. The intended interpretation of these symbols is multiplication in the algebra restricted to the sort S_n.
(4) For each $n \in \mathbb{N}$, there is a unary function symbol $*_n$ whose modulus of uniform continuity is $\delta(\epsilon) = \epsilon$. The intended interpretation of these symbols is for the adjoint restricted to each sort.
(5) For each $n \in \mathbb{N}$, there are two constant symbols 0_n and 1_n which lie in the sort S_n. The intended interpretation of these symbols are the elements 0 and 1.
(6) For each $n \in \mathbb{N}$ and $\lambda \in \mathbb{C}$, there is a unary function symbol λ_n whose domain is S_n and range is S_{mn}, where $m = \lceil |\lambda| \rceil$, and with modulus of uniform continuity $\delta(\epsilon) = \frac{\epsilon}{|\lambda|}$. The intended interpretation of these symbols is scalar multiplication by λ restricted to S_n. (If one is interested in keeping the language countable and computable, one may restrict to scalars whose real and imaginary parts are rational.)
(7) For each $m, n \in \mathbb{N}$ with $m < n$, we have unary function symbols $\iota_{m,n}$ with domain S_m and range S_n and whose modulus of uniform continuity is $\delta(\epsilon) = \epsilon$. The intended interpretation of these symbols is the inclusion map between the sorts.
(8) For each $n \in \mathbb{N}$, we have a unary predicate symbol φ_n, whose range is $[-n, n]$ and whose modulus of uniform continuity is $\delta(\epsilon) = \frac{\epsilon}{\sqrt{2}}$. The intended interpretation of this symbol is the restriction of the state to S_n. Technically speaking, there should really be two such symbols, one for the real and imaginary parts of the states, but we content ourselves to abuse notation here.
(9) For each $n \in \mathbb{N}$, we have function symbols $\mathbf{P}_n : S_n \to S_{3n}$, $\mathbf{Q}_n : S_n \to S_{3n}$, and $\mathbf{R}_n : S_n \to S_{6n}$. The intended interpretations of these symbols is as in the Rieffel-Van Daele theory from Section 3 (see Corollary 3.6).

We associate to each W*-probability space (M, φ) in W^*-Prob an $\mathcal{L}_{W^*}$-structure $\mathcal{D}(M, \varphi)$, called its **dissection**, by interpreting the symbols above with their intended meanings as outlined when introducing the language.

The goal now is to axiomatize the class of such dissections. We consider the following $\mathcal{L}_{W^*}$-theory T_{W^*}:

(1) Axioms for a *-algebra.
(2) Axioms saying that φ is a positive linear functional.
(3) Axioms saying the connecting maps preserve addition, multiplication, adjoints and φ.
(4) Axioms saying that the metric is derived from $\|\cdot\|_\varphi^\#$.

(5) Axioms requiring that the elements of S_n are totally n-bounded. More precisely, for each $n, k \in \mathbb{N}$, we have the axioms

$$\sup_{x \in S_n} \left(\sup_{y \in S_k} \max\{\varphi((xy)^*xy) \dot{-} n\varphi(y^*y), \varphi((yx)^*yx) \dot{-} n\varphi(y^*y)\} \right)$$

and similar axioms with x replaced by x^*.

(6) Axioms expressing that the inclusion maps are isometries and have the correct ranges. Specifically, for each n, and for all k we add:

$$\sup_{x \in S_1} \big| \|\iota_n(x)\| - \|x\| \big|$$

and

$$\sup_{x \in S_1} \sup_{z \in S_k} \big| \varphi((zx)^*(zx) - \varphi((z\iota_n(x))^*((z\iota_n(x))) \big|.$$

(7) Axioms expressing that the inclusion $\iota_n : S_1 \to S_n$ has range consisting of all $x \in S_n$ with x totally 1-bounded:

$$\begin{aligned} \sup_{x \in S_n} \sup_{z \in S_k} \inf_{y \in S_1} \max\big\{ &\|x - \iota_n(y)\| \dot{-} (\|x\| \dot{-} 1), \\ &\|x - \iota_n(y)\| \dot{-} \big(\varphi((zx)^*(zx)) \dot{-} 1\,\varphi(z^*z)\big), \\ &\|x - \iota_n(y)\| \dot{-} \big(\varphi((zx^*)^*(zx^*)) \dot{-} 1\,\varphi(z^*z)\big)\big\}. \end{aligned}$$

(8) Axioms stating that $\mathbf{P}_n$ is the orthogonal projection on to the set of self-adjoint elements. This is taken care of by the following two axioms:

$$\sup_{x \in S_n} d_{3n}(\mathbf{P}_n(x), \mathbf{P}_n(x)^*)$$

and

$$\sup_{x \in S_n} \sup_{y \in S_1} \varphi((y + y^*)(x - \mathbf{P}_n(x))).$$

(9) Axioms expressing the relation between $\mathbf{P}_n$ and $\mathbf{Q}_n$:

$$\sup_{x \in S_n} d_{3n}(\mathbf{Q}_n(x), i\mathbf{P}_n(-ix)).$$

(10) Axiom expressing the relation between $\mathbf{P}_n$, $\mathbf{Q}_n$, and $\mathbf{R}_n$:

$$\sup_{x \in S_n} d_{6n}(\mathbf{R}_n(x), \mathbf{P}_n(x) + \mathbf{Q}_n(x)).$$

(11) For each $a \in \mathbb{N}$, axioms that expresses that the model is closed under $h_a(\log \Delta)$. By Lemma 4.2, this can be taken care of with the following axioms:

$$\sup_{x \in S_K} \inf_{y \in S_K} d(2\mathbf{R}_n(2 - \mathbf{R}_n)x, (e^{-a}(2 - \mathbf{R}_n)^2 + e^a\mathbf{R}_n^2)y).$$

Note that in some of the axioms we are being a bit careless. For example, in the last axiom, instead of x and y we should be considering their images under suitable embeddings i and likewise the metric d should have a subscript corresponding to the appropriate sort.

We also point out that the theory T_{W^*} is effective (that is to say, computably enumerable).

Theorem 5.1. *T_{W^*} axiomatizes the class of dissections of W^*-probability spaces. Moreover, the class of models of T_{W^*} is categorically equivalent to the category of W^*-probability spaces.*

Proof. Based on the calculations in Section 3, it is straightforward to see that, for any W^*-probability space (M, φ), the dissection $\mathcal{D}(M, \varphi)$ is a model of T_{W^*}.

Conversely, consider a model A of the theory T_{W^*}. We begin by forming the direct limit M_0 of the sorts $S_n(A)$ for $n \in \mathbb{N}$ via the embeddings i. Using the interpretation of the function symbols on each sort, we see that M_0 is naturally a complex $*$-algebra. Furthermore, using the predicate for the state on each sort, one can define an inner product $\langle x, y\rangle := \varphi(y^*x)$ on M_0. We let H denote the Hilbert space completion of M_0 with respect to this inner product. Note that, for $a \in S_n(A)$, we have that $\varphi(a) = \langle a, 1\rangle = \langle \pi(a)1, 1\rangle$. For each $a \in M_0$, the maps $b \mapsto ab : M_0 \to M_0$ and $b \mapsto ba : M_0 \to M_0$ extend to bounded linear operators $\pi(a) : H \to H$ and $\pi'(a) : H \to H$ respectively satisfying $\|\pi(a)\|, \|\pi'(a)\| \leq n$ if $a \in S_n(A)$.

We let M_A denote the strong closure of $\pi(M_0)$ inside of $\mathcal{B}(H)$ and we let φ_A denote the faithful, normal state on M given by $\varphi(x) = \langle x1, 1\rangle$ for all $x \in M$; by the previous paragraph, φ_A agrees with φ on M_0 once M_0 is identified with its image via π.

We claim that $\pi(M_0) \subseteq B(H)$ is a good subalgebra of M_A, whence, by Theorem 4.3, $\pi(M_0) = M_{\mathrm{tb}}$. By definition, $\pi(M_0)$ is a strongly closed $*$-subalgebra of M_A. Moreover, by the strong density of $\pi(M_0)$ in M_A, it is clear that $\pi(M_0) \subseteq (M_A)_{\mathrm{tb}}$ and, moreover, if $a \in S_n(A)$, then $\pi(a) \in S_n(M_0)$. Now suppose that $(\pi(x_m))_{m\in\mathbb{N}}$ is a $\|\cdot\|^{\#}_{\varphi_A}$-Cauchy sequence from $S_1(\pi(M_0))$. By axiom (7), we can write $x_m = i(y_m)$, where $y \in S_1(A)$ and i is an appropriate embedding; by completeness of $S_1(A)$, it follows that $(y_m)_{m\in\mathbb{N}}$ has a $\|\cdot\|^{\#}_{\varphi}$ limit in $S_1(A)$ and hence $(\pi(x_m))_{m\in\mathbb{N}}$ has a $\|\cdot\|^{\#}_{\varphi_A}$ limit in $S_1(\pi(M_0))$. Finally, $\pi(M_0)$ is closed under $h_a(\log(\Delta))$ for all $a \in \mathbb{N}$ by axioms (8)-(11).

At this point, we have shown that, to a model A of T_{W^*}, we can associate a $*$-algebra M_0 of operators on a Hilbert space H and a faithful $*$-representation $\pi : M_0 \to \mathcal{B}(H)$ such that, setting M_A to be the strong closure of $\pi(M_0)$ and letting φ_A denote the vector state on M_A corresponding to the identity element of M_0,

we have $\mathcal{D}(M_A, \varphi_A) = A$. In particular, if we start with a W*-probability space (M, φ) and let $A := \mathcal{D}(M, \varphi)$, we have (using that the totally bounded vectors in M are dense in M) that $(M_A, \varphi_A) = (M, \varphi)$ and $\mathcal{D}(M_A, \varphi_A) = \mathcal{D}(M, \varphi)$.

Finally, we observe that embeddings between W*-probability spaces correspond to embeddings between the corresponding dissections. One direction of this claim is obvious; to see the other, suppose that $\mathcal{D}(M, \varphi) \subseteq \mathcal{D}(N, \psi)$. By Lemma 2.8 above, we must show that, for each $a \in M$, the $\|\cdot\|_\varphi$-distance between a and the self-adjoint elements of M is the same as the $\|\cdot\|_\psi$-distance between a and the self-adjoint elements of N. However this follows from the fact that the embedding preserves the interpretation of $\mathbf{P}$ and the density of bounded elements. □

Remark 5.2. The above theory T_{W^*} consists of ∀∃-axioms (see axioms (7) and (12)) whereas Dabrowski's theory is universal. We believe that the increase in quantifier complexity of the axiomatization is justified by having access to full multiplication.

6. Connection with the Ocneanu ultraproduct

In this section, we show that the pre-existing ultraproduct construction for W*-probability spaces, known as the Ocneanu ultraproduct, corresponds to the model-theoretic ultraproduct via the equivalence of categories proven in the previous section. We begin by recalling the construction of the Ocneanu ultraproduct.

Given a family $(M_i, \varphi_i)_{i \in I}$ of W*-probability spaces and an ultrafilter $\mathcal{U}$ on I, we set

$$\mathcal{I} := \mathcal{I}(M_i, \varphi_i) := \{(m_i) \in \ell^\infty(M_i, I) : \lim_{\mathcal{U}} \|m_i\|_{\varphi_i}^\# = 0\}$$

and set

$$\mathcal{M} := \mathcal{M}(M_i, \varphi_i) := \{(m_i) \in \ell^\infty(M_i, I) : (m_i)\mathcal{I} + \mathcal{I}(m_i) \subset \mathcal{I}\}.$$

By construction, $\mathcal{I}$ is a closed two-sided ideal of $\mathcal{M}$. The **Ocneanu ultraproduct** is the quotient $\prod^{\mathcal{U}} M_i := \mathcal{M}/\mathcal{I}$. The resulting C*-algebra is a σ-finite von Neumann algebra with faithful normal state given by $\varphi^{\mathcal{U}} := \lim_{\mathcal{U}} \varphi_i$. Given $(m_i) \in \mathcal{M}$, we write $(m_i)^\star$ for its equivalence class in $\prod^{\mathcal{U}} M_i$. For use later on, we record the well-known fact that if $m \in \prod^{\mathcal{U}} M_i$, then we may find $(m_i) \in \mathcal{M}$ such that $\|m_i\| = \|m\|$ for all $i \in I$; this is a consequence of the observation that $\prod^{\mathcal{U}} M_i$ is the quotient of the C*-algebra $\mathcal{M}$ by the ideal of sequences which go to 0 in norm along $\mathcal{U}$.

Meanwhile, for a family of $\mathcal{L}_{W^*}$-structures $(A_i)_{i \in I}$ and an ultrafilter $\mathcal{U}$ on I, we let $\prod_{\mathcal{U}} A_i$ denote the model-theoretic ultraproduct, which is once again an $\mathcal{L}_{W^*}$-structure. For an element $\prod_{i \in I} A_i$, we let $(a_i)^\bullet$ denote its equivalence class in $\prod_{\mathcal{U}} A_i$. The main result of this section is the following:

Theorem 6.1. *Suppose that (M_i, φ_i) is a family of* W*-probability spaces for all $i \in I$ and $\mathcal{U}$ is an ultrafilter on I. Then $\mathcal{D}(\prod^{\mathcal{U}}(M_i, \varphi_i)) \cong \prod_{\mathcal{U}} \mathcal{D}(M_i, \varphi_i)$.*

Before we prove the previous theorem, we need a few preliminary facts about *spectral subspaces*. Given a W*-probability space (M, φ) and $a > 0$, one can define the spectral subspace $M(\sigma^{\varphi}, [-a, a])$, which is a particular subset of M; see [1, Section 2.2] for the precise definition. We simply record the facts about spectral subspaces that we need here:

Facts 6.2.

(1) $M(\sigma^{\varphi}, [-a, a])$ is closed under adjoints.
(2) If (M_i, φ_i) is a family of W*-probability spaces, then a sequence $(m_i) \in \ell^{\infty}(M_i, I)$ belongs to $\mathcal{M}(M_i, \varphi_i)$ if and only if: for every $\epsilon > 0$, there is $(x_i) \in \mathcal{M}(M_i, \varphi_i)$ and $a > 0$ such that each $x_i \in M_i(\sigma^{\varphi_i}, [-a, a])$, $\lim_{\mathcal{U}} \|x_i - a_i\|^{\#}_{\varphi^{\mathcal{U}}} < \epsilon$, and $\|(y_i)^{\star}\| \leq \|(x_i)^{\star}\|$.*
(3) If $x \in M(\sigma^{\varphi}, [-a, a])$ for some $a > 0$, then the map $t \mapsto \sigma^{\varphi}_t(x)$ extends to an entire function $\mathbb{C} \to M$; moreover, for any $z \in \mathbb{C}$, there is a constant $C_{a,z}$ depending only on a and z such that $\|\sigma^{\varphi}_z(x)\| \leq C_{a,z}\|x\|$.
(4) If $x \in M(\sigma^{\varphi}, [-a, a])$, then $x \in M_{\mathrm{tb}}$ and $\|x\|_{\mathrm{right}}, \|x^*\|_{\mathrm{right}} \leq C_{a,i/2}\|x\|$.

Proof. Item (1) follows from the definition; see also [1, Section 2.2]. Item (2) is [1, Proposition 4.11]. Item (3) can be found in the proof of [1, Lemma 4.13]. Item (4) is explicit in [1, Lemma 4.13], but we include the proof for the sake of completeness. Suppose $x \in M(\sigma^{\varphi}, [-a, a])$ and take $y \in M$; we show that $\|yx\omega\| \leq C_{a,i/2} \cdot \|x\| \cdot \|y\omega\|$. To see this, we calculate as follows:

$$\begin{aligned}
\|yx\omega\| &= \|(JyJ)(S\Delta^{-1/2}x\omega)\| \\
&= \|(JyJ)S\sigma^{\varphi}_{i/2}(x)\omega)\| \\
&\leq \|(\sigma^{\varphi}_{i/2}(x))^*(JyJ)(\omega)\| \\
&\leq \|\sigma_{i/2}(x)^*\| \cdot \|(JyJ)(\omega)\| \\
&\leq C_{a,i/2} \cdot \|x\| \cdot \|y\omega\|.
\end{aligned}$$

The result for x^* is now immediate from (1). □

Proof of Theorem 6.1. First suppose that $(a_i)^{\bullet} \in \prod_{\mathcal{U}} S_n(M_i)$. It is clear that $(a_i) \in \ell^{\infty}(M_i, I)$; we wish to show that $(a_i) \in \mathcal{M}$. To see this, suppose that $(m_i) \in \mathcal{I}$; we show that $(a_i m_i), (m_i a_i) \in \mathcal{I}$. Suppose, towards a contradiction, that $(a_i m_i) \notin \mathcal{I}$ and set $L := \lim_{\mathcal{U}} \|a_i m_i\|^{\#}_{\varphi_i} \neq 0$. Consider the set of $i \in I$ for which $\|m_i\|^{\#}_{\varphi_i} < L/2n$, which belongs to $\mathcal{U}$ since $(m_i) \in \mathcal{I}$. For a $\mathcal{U}$-large subset of these i, we have that $\|a_i m_i\|^{\#}_{\varphi_i} > L/2$; for these i, we see that the operator

norm of a_i is greater than n or the right norm of a_i^* is greater than n, which is a contradiction. To see this, assuming $\|a_i\|, \|a_i^*\|_{\text{right}} \leq n$, we have

$$\|a_i m_i\|_{\varphi_i}^{\#} = \sqrt{\frac{\|a_i m_i\|_{\varphi_i}^2 + \|m_i^* a_i^*\|_{\varphi_i}^2}{2}} \leq \sqrt{\frac{\|a_i\|^2 \|m_i\|_{\varphi_i}^2 + \|a_i^*\|_{\text{right}}^2 \|m_i^*\|_{\varphi_i}^2}{2}} \leq n\|m_i\|_{\varphi_i}^{\#},$$

as claimed. The proof that $(m_i a_i) \in \mathcal{I}$ proceeds similarly, using that each a_i is right n-bounded and a_i^* is left n-bounded. It is now clear that $(a_i)^\bullet = (a_i)^\star$ and that this element is in $S_n(\prod^{\mathcal{U}}(M_i, \varphi_i))$.

By Theorem 5.1, we now have that the von Neumann algebra associated to $\prod_{\mathcal{U}} \mathcal{D}(M_i, \varphi_i)$ embeds into $\prod^{\mathcal{U}}(M_i, \varphi_i)$; it suffices to show that this embedding is actually onto. To see this, it suffices to show that, given any $x \in \prod^{\mathcal{U}}(M_i, \varphi_i)$ with $\|x\| = 1$ and any $\epsilon > 0$, there is $n > 0$ and $y \in \prod_{\mathcal{U}} S_n(M_i)$ such that $\|x - y\|_{\varphi^{\mathcal{U}}}^{\#} < \epsilon$. Write $x = (x_i)^\star$ and take $y = (y_i)^\star$ as in Fact 6.2(2) for some $a > 0$. We show that $y \in \prod_{\mathcal{U}} S_n(M_i)$ for some $n > 0$. Indeed, as mentioned above, we may assume that $\|y_i\| \leq 1$ for all $i \in I$ whence, by Fact 6.2(4), we have that $\|y_i\|_{\text{right}}, \|y_i^*\|_{\text{right}} \leq C_{a,i/2}$ for all $i \in I$ and thus $y \in \prod_{\mathcal{U}} S_n(M_i)$ when $n \geq C_{a,i/2}$. □

In the rest of the paper, we follow model-theoretic conventions and let $\prod_{\mathcal{U}}(M_i, \varphi)$ (or simply $\prod_{\mathcal{U}} M_i$ if the states are clear from context) denote the Ocneanu ultraproduct.

7. Expanding the language by the modular automorphism group

In this section, we show how to extend the theory T_{W^*} to a theory $T_{W^*-\text{mod}}$ in a language $\mathcal{L}_{W^*-\text{mod}}$ extending $\mathcal{L}_{W^*}$ which captures the modular automorphism group of the state. That such an extension should be possible is suggested by the following result, which is essentially [1, Theorem 4.1]. (In [1], the result is only stated for ultrapowers, but the proof goes through *mutatis mutandis* for ultraproducts.)

Fact 7.1. *Suppose that $(M_i, \varphi_i)_{i \in I}$ is a family of* W**-probability spaces and $\mathcal{U}$ is an ultrafilter on I. Set $(M, \varphi) = \prod_{\mathcal{U}}(M_i, \varphi_i)$. Then, for any $t \in \mathbb{R}$ and $(x_i)^\bullet \in M$, we have*

$$\sigma_t^{\varphi}(x_i)^\bullet = (\sigma_t^{\varphi_i}(x_i))^\bullet.$$

An immediate consequence of the previous result, by Beth's definability theorem (see, for example, [5]), is the following:

Corollary 7.2. *For all $t \in \mathbb{R}$, σ_t is a T_{W^*}-definable function. Moreover, if φ_t is a T_{W^*}-definable predicate defining σ_t, then the map $t \mapsto \varphi_t$ is continuous with respect to the logic topology.*

By the previous corollary, we can add symbols for the modular automorphism group and know that there is a definitional expansion $T_{W^*-\mathrm{mod}}$ of T_{W^*} in this larger language that captures the fact that these new symbols are indeed interpreted as the modular automorphism group. We now proceed to show how we can give concrete, effective axioms for $T_{W^*-\mathrm{mod}}$.

First we show that Δ_φ^{it} preserves our sorts. We recall that if $a \in M_{\mathrm{tb}}$, then $\pi'_\varphi(a) \in M'$, whence $J_\varphi \pi'_\varphi(a) J_\varphi \in M$ by Fact 3.7.

Lemma 7.3. *Suppose that $a \in M_{\mathrm{tb}}$. Then $\|J_\varphi \pi'_\varphi(a) J_\varphi\| \leq \|a\|_{\mathrm{right}}$.*

Proof. Fix $b \in M'$ and set $d := J_\varphi b J_\varphi \in M$. Then $J_\varphi(b\omega) = d\omega$ and so

$$\|(J_\varphi \pi'_\varphi(a) J_\varphi)(b\omega)\|_\varphi = \|\pi'_\varphi(a) d\omega\|_\varphi \leq \|a\|_{\mathrm{right}} \|d\omega\|_\varphi = \|a\|_{\mathrm{right}} \|b\omega\|_\varphi$$

since J_φ is an isometry. □

Proposition 7.4. *Suppose that $a \in M_{\mathrm{tb}}$. Then for all $t \in \mathbb{R}$, we have $\Delta_\varphi^{it}(a\omega) = b\omega$ for unique $b \in M_{\mathrm{tb}}$ with $\|b\|_{\mathrm{right}} = \|a\|_{\mathrm{right}}$.*

Proof. By Fact 3.8, we know that $\Delta_\varphi^{it}(a\omega) = b\omega$ for a unique $b \in M$. Since $a \in M_{\mathrm{tb}}$, we have that $J_\varphi \pi'_\varphi(a) J_\varphi = \pi_\varphi(c)$ for some $c \in M$. By Lemma 7.3 above, we have that $\|c\| \leq \|a\|_{\mathrm{right}}$. It follows that $J_\varphi a\omega = J_\varphi \pi'_\varphi(a) J_\varphi \omega = c\omega$. Consequently, by Fact 3.8 above, for any $d \in M$, we have

$$(\Delta_\varphi^{it} J_\varphi \pi_\varphi(c) J_\varphi \Delta_\varphi^{-it})(d\omega) = \pi_\varphi(d)(\Delta_\varphi^{it}(J_\varphi c\omega)) = \pi_\varphi(d)(b\omega).$$

Taking $\|\cdot\|_\varphi$ and using that Δ^{it} and J are isometries, we have

$$\|\pi_\varphi(d)(b\omega)\|_\varphi \leq \|\pi_\varphi(c)\| \|d\omega\|_\varphi \leq \|a\|_{\mathrm{right}} \|d\omega\|_\varphi.$$

It follows that $\|b\|_{\mathrm{right}} \leq \|a\|_{\mathrm{right}}$. By applying Δ^{-it}, we achieve equality. □

The previous proposition and the remarks after Fact 3.8 immediately imply:

Corollary 7.5. *If $a \in S_n(M)$, then for all $t \in \mathbb{R}$, we have $\Delta^{it}(a\omega) = b\omega$ for a unique $b \in S_n(M)$.*

Let $X \subseteq (0, 2)$ denote the spectrum of R. For $t \in \mathbb{R}$, we let $f_t : X \to \mathbb{C}$ be the function defined by $f_t(x) = x^{it}$. Take polynomial functions $f_{t,n}$ on X with coefficients from $\mathbb{Q}(i)$ such that $\|f_t - f_{t,n}\|_\infty < \frac{1}{n}$. Moreover, these polynomials can be found effectively in the sense that the map which upon (t, n) returns the coefficients of $f_{t,n}$ from $\mathbb{N}^2$ to $\mathbb{Q}(i)^{<\omega}$ is a computable map.

The following lemma is straightforward from the definition of Δ^{it}:

Lemma 7.6. *For all $m, n \geq 1$, we have*

$$\|\Delta^{it} - f_{t,m}(2 - R) f_{-t,n}(R)\| < \frac{1}{m}\|f_{-t}\|_\infty + \frac{1}{n}\|f_{t,m}\|_\infty.$$

Denoting the quantity on the right hand side of the inequality appearing in the previous lemma by $\delta_{t,m,n}$, we note that the map $(t,m,n) \mapsto \delta_{t,m,n}$ is computable. We also set $q_{t,n} \in \mathbb{N}$ to an integer such that $(f_{-t,n}(R_1))(S_1) \subseteq S_{q_{t,n}}$. For each $t \in \mathbb{Q}$, we add a symbol $\mathbf{\Delta}^{it} : S_1 \to S_1$ to the language and continue our enumeration of $T_{W^*-\mathrm{mod}}$ by adding the following axioms to our theory:

(12) $\sup_{x \in S_1}[\|\mathbf{\Delta}^{it}(x) - f_{t,m}(2 - \mathbf{R}_{q_{t,n}})(f_{-t,n}(\mathbf{R}_1)(x)))\|_\varphi \ \dot{-}\ \delta_{t,m,n}]$.

We note that the above description of the axioms in (12) is a bit sloppy as the term involving the $\mathbf{R}$'s take values in some sort S_p with p predictably depending on t, m, and n, whence we should technically be plugging $\Delta^{it}(x)$ into an appropriate inclusion mapping.

One last lemma before we can complete our axiomatization; the proof follows immediately from Proposition 7.4.

Lemma 7.7. *For each $a \in S_n(M)$ and $t \in \mathbb{R}$, we have $\sigma_t^\varphi(a) \in S_n$.*

We are finally ready to complete the axiomatization $T_{W^*-\mathrm{mod}}$. We add function symbols σ_t to the language and add the following axioms:

(13) $\sup_{a,x \in S_1} d_1(\sigma_t(a)x, \mathbf{\Delta}^{it}(a \cdot \mathbf{\Delta}^{-it}(x)))$.

We have now described a language $\mathcal{L}_{W^*-\mathrm{mod}}$ extending the language $\mathcal{L}$ and an *effective* $\mathcal{L}_{W^*-\mathrm{mod}}$-theory $T_{W^*-\mathrm{mod}}$ extending T_{W^*} for which the following theorem holds:

Theorem 7.8. *The category of models of $T_{W^*-\mathrm{mod}}$ consists of the dissections of W*-probability spaces with the symbols $\mathbf{\Delta}^{\mathbf{it}}$ interpreted as in Section 3 and with the symbols σ_t interpreted as the modular automorphism group of the state (restricted to the sort S_1).*

8. Axiomatizable and local classes

In this section, we determine whether or not natural classes of W*-probability spaces are axiomatizable or local (defined below). A similar endeavour for the class of tracial von Neumann algebras was undertaken in [7].

We begin with a series of negative results; these results were also mentioned by Dabrowski in [4, Section 2.3] in his language, but we offer a few details here for the convenience of the reader.

The following is [1, Proposition 6.3]:

Fact 8.1. *There is a family φ_n of faithful normal states on the hyperfinite II_1 factor $\mathcal{R}$ such that $\prod_{\mathcal{U}}(\mathcal{R}, \varphi_n)$ is* not *semifinite.*

In what follows, given a property P of von Neumann algebras, we abuse terminology and speak of the class of algebras satisfying P when referring to the class of pairs W*-probability spaces (M, φ) where M satisfies property P. We also say that the class of algebras satisfying P is axiomatizable if the class of dissections of pairs (M, φ) with M satisfying P is an axiomatizable class.

Corollary 8.2. *The following classes are not elementary classes: finite algebras, semifinite algebras, II_1 factors.*

Proposition 8.3. *The class of II_∞ factors is also not axiomatizable.*

Proof. Let M be a II_1 factor with unique trace τ and set $N := M \otimes B(\ell^2)$. Let (e_i) be the standard orthonormal basis for ℓ^2. Define states φ_n on N by setting $\varphi_n(x \otimes T) := \tau(x)\left(\sum_i \lambda_{n,i} \langle Te_i\,,\, e_i\rangle\right)$ with $\lambda_{n,i} \neq 0$; note that each φ_n is faithful. Further assume that $\lambda_{n,1} \to 1$ while the rest tend to zero. Then $\prod_{\mathcal{U}}(N, \varphi_n) \cong (M, \tau_M)^{\mathcal{U}}$, a II_1 factor. □

Remark 8.4. In [4, Theorem 2.10], it is shown that the class of W*-probability spaces of the form $(N \otimes \mathcal{B}(H), \varphi)$, where N is a tracial von Neumann algebra and φ is a *geometric state* on $N \otimes \mathcal{B}(H)$ is elementary in a natural expansion of Dabrowski's language by constants for matrix units. Here, a geometric state on $N \otimes \mathcal{B}(H)$ means that the restriction of the state to $\mathcal{B}(H)$ has a particularly simple form. It is also shown that, in the same expansion, the subclasses of type I_∞ factors and type II_∞ factors (again equipped with geometric states) are each axiomatizable. The interested reader can readily adapt this axiomatization to our setting.

As pointed out at the end of Section 3.1 of [1], given a von Neumann algebra M and two faithful normal states φ_1 and φ_2, we have $\prod_{\mathcal{U}}(M, \varphi_1) \cong \prod_{\mathcal{U}}(M, \varphi_2)$, whence there is a well-defined notion of the **ultrapower** of M. In this case, we denote the algebra simply by $M^{\mathcal{U}}$. The following is [1, Theorem 6.18]:

Fact 8.5. *Suppose that M is a σ-finite factor of type III_0. Then $M^{\mathcal{U}}$ is* not *a factor.*

Recall that a class of structures is called **local** if it is closed under elementary equivalence; equivalently, the class is closed under isomorphism, ultraroot, and ultrapower.

Corollary 8.6. *The following classes are not local: factors, type III_0 factors.*

However:

Proposition 8.7. *The following classes are local: finite algebras, semi-finite algebras, type I_n factors (for a fixed n), type II_1 factors, type II_∞ factors.*

Proof. The only classes that need explanation are that of the case of semi-finite algebras and type II_∞ factors; these classes are seen to be local by using the fact (which appears to be folklore, though the interested reader can find a proof in [12, Lemma 2.8]) that, given a von Neumann algebra M with separable predual, a separable Hilbert space H, and an ultrafilter $\mathcal{U}$ on $\mathbb{N}$, that $(M\overline{\otimes}B(H))^{\mathcal{U}} \cong M^{\mathcal{U}}\overline{\otimes}B(H)$. □

Finally, we mention the following positive result.

Proposition 8.8. *Suppose that $S \subseteq (0,1]$ is closed. Then the set of type III_λ factors where $\lambda \in S$ is an axiomatizable class.*

Proof. In [1, Theorem 6.11], it is shown that if M is a type III_λ factor for $\lambda \in (0,1]$ and φ_n is any family of faithful normal states on M, then $\prod_{\mathcal{U}}(M, \varphi_n)$ is once again a type III_λ factor. The same proof shows that the class of pairs (M, φ), where M is a type III_λ factor with $\lambda \in S$, is closed under ultraproducts.

It remains to see that this class is closed under ultraroots. Thus, suppose that (M, φ) is such that $M^{\mathcal{U}}$ is a type III_λ factor with $\lambda \in S$. It is easy to see that M must then be a factor. M cannot be semifinite for then $M^{\mathcal{U}}$ would also be semifinite. M cannot be III_0 for then $M^{\mathcal{U}}$ would not be a factor by Fact 8.5. Thus, M must be type III_η for $\eta \in (0,1]$; by the previous paragraph, it must be that $\eta = \lambda$. □

Remarks 8.9.

(1) The previous theorem in the case that S is a singleton was stated by Dabrowski in his language in [4, Theorem 2.10(3)].
(2) In [9, Section 4], it was shown that the class of III_1 factors is $\forall\exists$-axiomatizable, whereas, for $\lambda \in (0,1)$, the class of III_λ factors is not $\forall\exists$-axiomatizable nor $\exists\forall$-axiomatizable, but is both $\forall\exists\forall$-axiomatiz-able and $\exists\forall\exists$-axiomatizable. In all cases, the quantifier-complexity of the axiomatizations are deduced from semantic criteria; it would be very interesting to give concrete (hopefully even effective) axiomatizations of these classes.
(3) In [4, Theorem 10(2)], Dabrowski showed that, for a fixed $\lambda \in (0,1)$, the class of W*-probability spaces of the form (M, φ), where M is a type III_λ factor and φ is a **periodic** state is axiomatizable in his language for W*-probability spaces. The interested reader can readily adapt Dabrowski's axiomatization to our setting.

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