SOBOLEV SPACES AND ELLIPTIC EQUATIONS

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ABSTRACT. Sobolev spaces play a critical role in the analysis of partial differential equations and their numerical approximations. In this chapter, we will provide a concise and informative overview of Sobolev spaces and the regularity theory of elliptic boundary value problems.

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1. ESSENTIAL FACTS FOR SOBOLEV SPACES

In this section, we present and explain main results (without proofs) on Sobolev spaces. We refer to [1] for comprehensive treatment of Sobolev spaces.

1.1. **Preliminaries.**

To establish the context for our discussion, we will introduce the fundamental components of Lipschitz domains, multi-index notation for differentiation, and a few basic functional spaces. These tools will serve as the foundation for our subsequent exploration and analysis.

Lipschitz domains. Throughout this section, we will focus primarily on bounded Lipschitz domains. A domain $\Omega \subset \mathbb{R}^n$ is said to be Lipschitz if its boundary $\partial\Omega$ can be locally represented as a Lipschitz continuous function. Specifically, for any $x \in \partial\Omega$, there exists a neighborhood $G \subset \mathbb{R}^n$ of x such that $G \cap \partial\Omega$ can be expressed as the graph of a Lipschitz continuous function under a suitable local coordinate system.

While all smooth domains are Lipschitz, many non-smooth domains also satisfy this property. For example, every polygonal domain in $\mathbb{R}^2$ or polyhedron in $\mathbb{R}^3$ is Lipschitz.

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Interestingly, every convex domain in $\mathbb{R}^n$ is also Lipschitz. However, certain domains such as two polygons touching at only one vertex are not Lipschitz.

Notation of Schwarz. Let $\alpha = (\alpha_1, \cdots, \alpha_n) \in \mathbb{Z}_+^n$, where $\mathbb{Z}_+$ is the set of non-negative integers, be a vector of nonnegative integers, denote

$$D^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdots x_n^{\alpha_n}} \quad \text{with sum } |\alpha| = \sum_{i=1}^n \alpha_i.$$

For a smooth function v and $x \in \mathbb{R}^n$, denote

$$D^\alpha v = \frac{\partial^{|\alpha|} v}{\partial x_1^{\alpha_1} \cdots x_n^{\alpha_n}}, \text{ and } x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}.$$

Some basic functional spaces. Several basic Banach spaces will often be used in this course. $C(\bar{\Omega})$ is the space of continuous functions on $\bar{\Omega}$ with the usual maximum norm

$$\|v\|_{C(\bar{\Omega})} = \max_{x \in \bar{\Omega}} |v(x)|.$$

$C_0^\infty(\Omega)$ denotes the space of infinitely differential functions in Ω that vanish in some neighborhood of $\partial\Omega$; namely any $v \in C_0^\infty(\Omega)$ satisfies $\operatorname{supp}(v) \subset \Omega$, where

$$\operatorname{supp}(v) = \text{closure of } \{x \in \Omega : \ v(x) \neq 0\}.$$

Let $E(\Omega)$ represent the equivalent class of Lebesgue integrable functions corresponding to the equivalence $u \sim v$ if $u = v$ almost everywhere. Given $1 \leq p < \infty$, we shall denote the usual Lebesgue space of pth power integrable functions by $L^p(\Omega) = \{v \in E(\Omega) : \int_\Omega |v|^p \, \mathrm{d}x < \infty\}$ and the essentially bounded function space by $L^\infty(\Omega)$. The norm is defined by

$$\|u\|_{p,\Omega} = \Big(\int_\Omega |u|^p \, \mathrm{d}x \Big)^{1/p} \quad \text{for } 1 \leq p < \infty,$$

and

$$\|u\|_{\infty,\Omega} = \operatorname*{ess\,sup}_\Omega |u|.$$

Exercise 1.1. *Prove that $L^p(\Omega), 1 \leq p \leq \infty$ defined above are Banach spaces.*

1.2. **Definition of Sobolev spaces.** Traditionally, we think of functions in terms of their pointwise values. However, this approach is insufficient when dealing with more general functions, such as distributions or generalized functions. To overcome this limitation, we define a function as a functional, which acts on a space of well-behaved test functions. By using this framework, we can extend the notion of functions to distributions, and the operation of "integration by parts" enables us to generalize the concept of differentiation to distributions.

We will first introduce the concept of Sobolev spaces for integer orders, using the framework of distributions and their weak derivatives. We will then define the fractional order Sobolev spaces by considering the pth power integrable of quotient of differences. Additionally, we will provide definitions of Sobolev spaces that satisfy certain zero boundary conditions.

Distributions and Weak Derivatives. We begin by introducing the function space $C_0^\infty(\Omega)$, which consists of infinitely differentiable functions with compact support in Ω. This space is a real vector space that can be equipped with a suitable topology, turning it into a topological vector space. We denote this space by $\mathcal{D}(\Omega)$, and a sequence of functions $\{\phi_k\} \subset C_0^\infty(\Omega)$ is said to converge to a function $\phi \in C_0^\infty(\Omega)$ in $\mathcal{D}(\Omega)$ if two conditions are satisfied:

(1) there exists a compact set $K \subset \Omega$ such that for all k, $\operatorname{supp}(\phi_k - \phi) \subset K$, and
(2) for every $\alpha \in \mathbb{Z}_+^n$, we have $\lim_{k\to\infty} \|D^\alpha(\phi_k - \phi)\|_\infty = 0$.

The space of all continuous linear functionals on $\mathcal{D}(\Omega)$ is called the distribution space, denoted by $\mathcal{D}'(\Omega)$. We equip $\mathcal{D}'(\Omega)$ with the weak-star topology. That is, in $\mathcal{D}'(\Omega)$, a sequence T_n converges to T in the distribution sense if and only if $\langle T_n, \phi\rangle \to \langle T, \phi\rangle$ for all $\phi \in \mathcal{D}(\Omega)$, where $\langle\cdot,\cdot\rangle : \mathcal{D}'(\Omega) \times \mathcal{D}(\Omega) \to \mathbb{R}$ is the duality pairing. A function $\phi \in \mathcal{D}(\Omega)$ is called a test function because the action of a distribution on ϕ can be interpreted as a test. The properties of a distribution can be extracted by choosing suitable test functions.

Example 1.2. By definition, an element in $\mathcal{D}'(\Omega)$ is uniquely determined by its action. The action could be very general and abstract as long as it is linear and continuous. As an example, let us introduce the Dirac delta distribution $\delta \in \mathcal{D}'(\Omega)$ with $0 \in \Omega \subseteq \mathbb{R}^n$

$$\langle \delta, \phi\rangle = \phi(0) \quad \text{for all } \phi \in \mathcal{D}(\Omega).$$

An essential class of distributions uses integration as the primary action. A function is said to be locally integrable if it is Lebesgue integrable over every compact subset of Ω and denote the corresponding space as $L^1_{\mathrm{loc}}(\Omega)$. We can embed $L^1_{\mathrm{loc}}(\Omega)$ into $\mathcal{D}'(\Omega)$ using integration as the duality action. For a function $u \in L^1_{\mathrm{loc}}(\Omega)$, we define the corresponding distribution $T_u \in \mathcal{D}'(\Omega)$ as

$$\langle T_u, \phi\rangle = \int_\Omega u\phi \,\mathrm{d}x \quad \text{for all } \phi \in \mathcal{D}(\Omega).$$

We shall still denote T_u by u. This correspondence between u and T_u is often used to identify an "ordinary" function as a distribution.

A distribution is also commonly referred to as a generalized function because the concept of distribution is more general than that of a classic function. One fundamental distribution that is not an "ordinary" function is the Dirac δ distribution introduced in Example 1.2. Indeed, one motivation for inventing the distribution space is to include the Dirac δ "function" in the general framework of function spaces.

Exercise 1.3. *Prove that it is not possible to represent the delta distribution by a locally integrable function.*

When u is a smooth function, integration by parts implies that, for any multi-index $\alpha \in \mathbb{Z}_+^n$ and any $\phi \in \mathcal{D}(\Omega)$,

$$\int_\Omega D^\alpha u(x)\phi(x)\ \mathrm{d}x = (-1)^{|\alpha|} \int_\Omega u(x) D^\alpha\phi(x)\ \mathrm{d}x.$$

Since ϕ has compact support in Ω, there are no boundary terms, and both sides of the equation are well-defined. This identity serves as the foundation for defining derivatives for a distribution. Specifically, if $T \in \mathcal{D}'(\Omega)$, then we define the weak derivative $D^\alpha T$ as the distribution given by

$$\langle D^\alpha T, \phi\rangle = (-1)^{|\alpha|}\langle T, D^\alpha\phi\rangle \quad \text{for all } \phi \in \mathcal{D}(\Omega).$$

It is easy to see that the weak derivative of a differentiable function coincides with its classical derivative. However, in general, the weak derivative is much weaker than the classical derivative, allowing differential operators to be extended from differential functions to a much larger space - the space of distributions. For example, we can talk about the derivatives of a discontinuous function.

Example 1.4. The Heaviside step function is defined as $S(x) = 1$ for $x > 0$ and $S(x) = 0$ for $x < 0$. By the definition

$$\int_{\mathbb{R}} S'\phi \, \mathrm{d}x = -\int_{\mathbb{R}} S\phi' \, \mathrm{d}x = -\int_0^\infty \phi' \, \mathrm{d}x = \phi(0).$$

Therefore $S' = \delta$ in the distribution sense but δ is not a function in $L^1_{\mathrm{loc}}(\Omega)$ (Exercise 1.3). Roughly speaking, any distribution is locally a (multiple) derivative of a continuous function. A precise version of this result can be found in Rudin [11].

The formal definition of distributions defines them as a subspace of a very large space. In general, no proper subset of the space of distributions contains all continuous functions and is closed under differentiation. This fact highlights the economy of the distribution extension of the function concept. A distribution is infinitely differentiable in the distribution sense.

Integer order Sobolev spaces. The Sobolev space of index (k, p), where k is a nonnegative integer and $p \geq 1$, is defined by

$$W^{k,p}(\Omega) \stackrel{\mathrm{def}}{=} \{v \in L^p(\Omega) : D^\alpha v \in L^p(\Omega) \quad \text{for all } \ |\alpha| \leq k\},$$

with a norm $\|\cdot\|_{k,p,\Omega}$ given by

$$\|v\|^p_{k,p,\Omega} \stackrel{\mathrm{def}}{=} \sum_{|\alpha|\leq k} \|D^\alpha v\|^p_{0,p(\Omega)}, \tag{1}$$

We will have occasions to use the seminorm $|\cdot|_{k,p,\Omega}$ given by

$$|v|^p_{k,p,\Omega} \stackrel{\mathrm{def}}{=} \sum_{|\alpha|=k} \|D^\alpha v\|^p_{0,p,\Omega}.$$

For $p = 2$, it is customary to write $H^k(\Omega) \stackrel{\mathrm{def}}{=} W^{k,2}(\Omega)$ which is a Hilbert space together with an inner product as follows

$$(u, v) = \sum_{|\alpha|\leq k} (D^\alpha u, D^\alpha v)$$

and the corresponding norm is denoted by $\|v\|_{k,\Omega} = \|v\|_{k,2,\Omega}$. The majority of our use of Sobolev space will be in this case.

Exercise 1.5. *Prove that $W^{k,p}(\Omega)$ is a Banach space.*

For a function in $L^p(\Omega)$, treating it as a distribution, its weak derivatives always exists as distributions, but may not be in the space $L^p(\Omega)$. An element in $W^{k,p}(\Omega)$ possesses certain smoothness.

Example 1.6. We consider the Heaviside function restricted to $(-1, 1)$ and still denote by S. The weak derivative of S is the delta distribution which is not integrable. Therefore $S \notin H^1(-1, 1)$.

Example 1.7. Let $u(x) = \mathrm{ReLU}(x) := \max\{0, x\}$ for $x \in (-1, 1)$. Then $u' = S$. Obviously $u \in L^2(-1, 1)$ and $u' \in L^2(-1, 1)$. Therefore $u \in H^1(-1, 1)$.

Example 1.6 and 1.7 explain that for a piecewise smooth function u to be in $H^1(\Omega)$ requires more global smoothness of u in Ω. See Exercise 1.18.

Fractional order Sobolev spaces. In the definition of the classic derivative, it takes the pointwise limit of the quotient of difference. For functions in Sobolev space, we shall use the pth power integrability of the quotient difference to characterize the differentiability.

For $0 < \sigma < 1$ and $1 \le p < \infty$, we define

$$W^{\sigma,p}(\Omega) = \left\{ v \in L^p(\Omega) : \int_\Omega \int_\Omega \frac{|v(x) - v(y)|^p}{|x-y|^{n+\sigma p}} \, \mathrm{d}x \, \mathrm{d}y < \infty \right\} \tag{2}$$

and

$$H^\sigma(\Omega) = W^{\sigma,2}(\Omega).$$

In $W^{\sigma,p}(\Omega)$, we define the following semi-norm

$$|v|^p_{\sigma,p,\Omega} \stackrel{\text{def}}{=} \int_\Omega \int_\Omega \frac{|v(x) - v(y)|^p}{|x-y|^{n+\sigma p}} \, \mathrm{d}x \, \mathrm{d}y, \tag{3}$$

and norm

$$\|v\|^p_{\sigma,p,\Omega} \stackrel{\text{def}}{=} \|v\|^p_{0,p,\Omega} + |v|^p_{\sigma,p,\Omega}. \tag{4}$$

In the definition of the fractional Sobolev space, the index of the dominator in the quotient of difference depends on the dimension of the space. This can be seen more clearly if we use polar coordinate in the integration. The fractional derivative seems weird. We now give a concrete example.

Example 1.8. For the Heavisde function S restricted to $(-1, 1)$, we look at the integral (3)) in the skewed coordinate $\tilde{x} = x - y, \tilde{y} = x + y$. To be in the space $W^{\sigma,p}(\Omega)$, it is essential to have the integral

$$\int_0^1 \frac{1}{\tilde{x}^{\sigma p}} \, \mathrm{d}\tilde{x} < \infty.$$

That is to require $\sigma < 1/p$. In particular, we conclude $S \in H^{1/2-\epsilon}(-1, 1)$ for any $0 < \epsilon \le 1/2$ but $S \notin H^{1/2}(-1, 1)$.

Given $s = k + \sigma$ with a real number $\sigma \in (0, 1)$ and an integer $k \ge 0$, define

$$W^{s,p}(\Omega) \stackrel{\text{def}}{=} \{v \in W^{k,p}(\Omega) : \ D^\alpha v \in W^{\sigma,p}(\Omega), |\alpha| \le k\}.$$

In $W^{s,p}(\Omega)$, we define the following semi-norm and norm

$$|v|_{s,p,\Omega} = \left(\sum_{|\alpha|=k} |D^\alpha v|^p_{\sigma,p,\Omega} \right)^{1/p}, \quad \|v\|_{s,p,\Omega} = \left(\sum_{|\alpha|\le k} \|D^\alpha v\|^p_{\sigma,p,\Omega} \right)^{1/p}.$$

Negative order Sobolev spaces. $W^{k,p}(\Omega)$ is a Banach space, i.e., it is complete in the topology induced by the norm $\|\cdot\|_{k,p,\Omega}$. Indeed $W^{k,p}(\Omega)$ is the closure of $C^\infty(\Omega)$ with respect to $\|\cdot\|_{k,p,\Omega}$. The closure of $C_0^\infty(\Omega)$ with respect to the same topology is denoted by $W_0^{k,p}(\Omega)$. For $p = 2$, we usually write $H_0^k(\Omega) = W_0^{k,2}(\Omega)$. Roughly speaking for $u \in H_0^1(\Omega)$, $u|_{\partial\Omega} = 0$ in an appropriate sense. Except $k = 0$ or $\Omega = \mathbb{R}^n$, $W_0^{k,p}(\Omega)$ is a proper subspace of $W^{k,p}(\Omega)$.

For $k \in \mathbb{N}$, $W^{-k,p}(\Omega)$ is defined as the dual space of $W_0^{k,p'}(\Omega)$, where p' is the conjugate of p, i.e., $1/p + 1/p' = 1$. In particular $H^{-k}(\Omega) = (H_0^k(\Omega))'$, and for $f \in H^{-1}(\Omega)$

$$\|f\|_{-1,\Omega} = \sup_{v \in H_0^1(\Omega)} \frac{\langle f, v \rangle}{\|v\|_{1,\Omega}}.$$

Since $C_0^\infty(\Omega)$ is dense in $W_0^{k,p}(\Omega)$, $W^{-k,p}(\Omega)$ can be thought as a subspace of $\mathcal{D}'(\Omega)$. Furthermore, we have the following characterization of $W^{-k,p}(\Omega)$ [11] which says, roughly speaking, $D^{-k}v \in L^p$ suppose we can define D^{-k} in an appropriate way.

Theorem 1.9. *Let $v \in \mathcal{D}'(\Omega)$. Then $v \in W^{-k,p}(\Omega)$ if and only if*

$$v = \sum_{|\alpha|\leq k} D^\alpha v_\alpha, \quad \text{for some } v_\alpha \in L^p(\Omega).$$

Note that $C_0^\infty(\Omega)$ is not dense in $W^{k,p}(\Omega)$ for $k \in \mathbb{N}$. Therefore the dual space of $W^{k,p}(\Omega)$ cannot be embedded as a subspace of the distribution space.

Example 1.10. Since the Heavside function $S \in L^2(-1,1)$, by Theorem 1.9, we have $\delta = S' \in H^{-1}(-1,1)$. But this is only true in one dimension; see Example 1.11.

Characterization of Sobolev spaces using Fourier transform. In the case when $\Omega = \mathbb{R}^n$, the Sobolev space $H^k(\mathbb{R}^n)$ can be characterized using the Fourier transform. We begin by defining $\mathcal{S}(\mathbb{R}^n)$ as the Schwartz functions, which consist of smooth functions with all derivatives decaying faster than any polynomial as the function approaches infinity. The Fourier transform

$$\hat{f}(\xi) = \frac{1}{(\sqrt{2\pi})^n} \int_{\mathbb{R}^n} f(x) e^{-2\pi i x\cdot\xi} \,\mathrm{d}x$$

is well defined for $f \in L^1(\mathbb{R}^n)$. However, if we restrict the transform to $\mathcal{S}(\mathbb{R}^n)$, the transform becomes an isomorphism. Furthermore, when we apply the transform to derivatives, we obtain a multiplication, i.e.,

$$\widehat{D^\alpha v} = (2\pi i\xi)^\alpha \hat{v},$$

which can be shown using integration by parts.

Given $v \in H^k(\mathbb{R}^n)$, by the Plancherel's theorem, we have

$$\|D^\alpha v\|_{0,\mathbb{R}^n} \eqsim \|\xi^\alpha \hat{v}\|_{0,\mathbb{R}^n}.$$

Therefore, by definition of the Sobolev norm,

$$\|v\|_{k,\mathbb{R}^n}^2 \eqsim \sum_{|\alpha|\leq k} \int_{\mathbb{R}^n} |\xi^\alpha|^2 |\hat{v}(\xi)|^2 \, d\xi.$$

Using the elementary inequalities

$$(1+|\xi|^2)^k \lesssim \sum_{|\alpha|\leq k} (|\xi|^2)^\alpha \lesssim (1+|\xi|^2)^k, \tag{5}$$

we conclude that

$$\|v\|_{k,\mathbb{R}^n} \eqsim \|(1+|\cdot|^2)^{k/2}\hat{v}\|_{0,\mathbb{R}^n}.$$

In the expansion of $(1+|\xi|^2)^{k/2}$, all terms involving $(|\xi|^2)^\alpha$, with $|\alpha| \leq k$, all included. This relation shows that the Sobolev space $H^k(\mathbb{R}^n)$ may be equivalently defined as

$$H^k(\mathbb{R}^n) = \{v \in L^2(\mathbb{R}^n), (1+|\xi|^2)^{k/2}\hat{v} \in L^2(\mathbb{R}^n)\}.$$

This alternative definition can be used to characterize Sobolev spaces with real index. Namely, for any given $s \in [0,\infty)$, the Sobolev space $H^s(\mathbb{R}^n)$ can be defined as follows:

$$H^s(\mathbb{R}^n) \stackrel{\text{def}}{=} \{v \in L^2(\mathbb{R}^n), (1+|\xi|^2)^{s/2}\hat{v} \in L^2(\mathbb{R}^n)\} \quad \text{for all } s \in [0,\infty) \tag{6}$$

with a norm defined by

$$\|v\|_{s,\mathbb{R}^n} \stackrel{\text{def}}{=} \|(1+|\cdot|^2)^{s/2}\hat{v}\|_{0,\mathbb{R}^n}. \tag{7}$$

Example 1.11. In $\mathbb{R}^n$, the Fourier transform of δ-distribution is one. By looking at the integration in the polar coordinate, we see $\delta \in H^{-s}(\mathbb{R}^n)$ for $s > n/2$ only. Therefore for $n \geq 2$, the delta distribution is not in $H^{-1}(\Omega)$.

1.3. **Extension theorems.** The extension theorem presented below is a fundamental result for Sobolev spaces. While Fourier transform is a powerful tool, unfortunately it only works for functions defined in the entire space $\mathbb{R}^n$. To extend results proved on the whole $\mathbb{R}^n$ to a bounded domain Ω, we can try to extend the function defined in $W^{k,p}(\Omega)$ to $W^{k,p}(\mathbb{R}^n)$. The extension of a function $u \in L^p(\Omega)$ is trivial; for example, we can simply set $u(x) = 0$ when $x \notin \Omega$, which is called zero extension. However, such trivial extensions will create a bad discontinuity along the boundary and thus cannot control the norm of derivatives, especially when the boundary is non-smooth. The extension of Sobolev space $W^{k,p}(\Omega)$ requires more sophisticated methods. We present the result below.

Theorem 1.12. *For any bounded Lipschitz domain Ω, for any $s \geq 0$ and $1 \leq p \leq \infty$, there exits a linear operator $E : W^{s,p}(\Omega) \to W^{s,p}(\mathbb{R}^n)$ such that*

(1) $Eu|_\Omega = u$, and
(2) E is continuous. More precisely there exists a constant $C(s, \Omega)$ which is increasing with respect to $s \geq 0$ such that, for all $1 \leq p \leq \infty$,

$$\|Ev\|_{s,p,\mathbb{R}^n} \leq C(s, \Omega)\|v\|_{s,p,\Omega} \quad \text{for all } v \in W^{s,p}(\Omega).$$

Theorem 1.12 is a well-known result for integer-order Sobolev spaces defined on smooth domains, and its proof can be found in most textbooks on Sobolev spaces. However, for Lipschitz domains and especially for fractional-order spaces, Theorem 1.12 is less well-known, and its proof for these cases is quite complicated. For integer-order Sobolev spaces on Lipschitz domains, Calder'on [4] and Stein [12] provide detailed treatments of the topic. For fractional-order Sobolev spaces on Lipschitz domains, McLean's book [10] is a valuable reference.

1.4. **Embedding theorems.** Embedding theorems are what make Sobolev spaces interesting and important. Sobolev spaces $W^{k,p}(\Omega)$ are defined using weak derivatives, which give a notion of smoothness weaker than that given by classical derivatives. Sobolev embedding theorems connect the concepts of smoothness using weak and classical derivatives. In essence, these theorems state that if a function is weakly smooth enough, it implies a certain degree of classical smoothness.

Example 1.13. In two dimensions, consider the function $u(x) = \ln|\ln|x||$ when $|x| < 1/e$ and $u(x) = 0$ when $|x| \geq 1/e$. It is easy to verify that $u \in H^1(\mathbb{R}^2)$. But u is unbounded, i.e., $u \notin C(\mathbb{R}^2)$.

Exercise 1.14. *Prove that if $u \in W^{1,1}(0, 1)$, then $u \in L^\infty(0, 1)$.*

We now present the general embedding theorem. For two Banach spaces B_1, B_0, we say B_1 is continuously embedded into B_0, denoted by $B_1 \hookrightarrow B_0$, if for any $u \in B_1$, it is also in B_0 and the embedding map is continuous, i.e., for all $u \in B_1$

$$\|u\|_{B_0} \lesssim \|u\|_{B_1}.$$

Theorem 1.15 (General Sobolev embedding)**.** *Let $1 \leq p \leq \infty, k \in \mathbb{Z}_+$ and Ω be a bounded Lipschitz domain in $\mathbb{R}^n$.*

Case 1. $kp > n$

$$W^{k,p}(\Omega) \hookrightarrow C(\bar{\Omega}).$$

Case 2. $kp = n$

$$W^{k,p}(\Omega) \hookrightarrow L^q(\Omega), \quad \textit{for all } q \in [1, \infty).$$

Furthermore

$$W^{n,1}(\Omega) \hookrightarrow C(\bar{\Omega}).$$

Case 3. $kp < n$

$$W^{k,p}(\Omega) \hookrightarrow L^q(\Omega), \quad \textit{with } \frac{1}{q} = \frac{1}{p} - \frac{k}{n}.$$

Exercise 1.16. *Use the Fourier transform to prove that if $u \in H^s(\mathbb{R}^n)$ for $s > n/2$, then $u \in L^\infty(\mathbb{R}^n)$. Then use the extension theorem to prove similar results for Lipschitz domains Ω.*

We can visualize Sobolev spaces in the upper right quadrant of a plane, where the x-axis corresponds to L^p spaces with L^p identified with $x = 1/p$, not $x = p$. The y-axis corresponds to the order of smoothness. For example, the point $(1/p, k)$ represents the Sobolev space $W^{k,p}(\Omega)$. The line passing through $(1/p, 0)$ with slope n (the dimension of Euclidean spaces) is the demarcation line for embeddings of Sobolev spaces into $L^p(\Omega)$ (see Figure 1). Any Sobolev space with indices corresponding to a point above that line is embedded into $L^p(\Omega)$. This visualization is adapted from DeVore [6] (page 93) and provides a simple way to keep track of various results and add to our understanding.

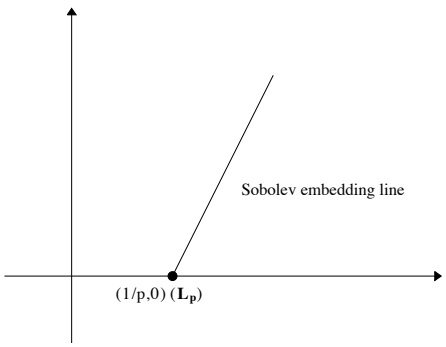


FIGURE 1. Sobolev embedding

To quickly determine whether a point lies above or below the demarcation line, we introduce the *Sobolev number*:

$$\mathrm{sob}_n(k, p) = k - \frac{n}{p}.$$

If $\mathrm{sob}_n(k, p) > 0$, functions from $W^{k,p}(\Omega)$ are continuous (or, more precisely, can be represented by a continuous function in its equivalence class). In general, we have the embedding result:

$$W^{k,p}(\Omega) \hookrightarrow W^{l,q}(\Omega) \quad \text{if } k > l \text{ and } \mathrm{sob}_n(k, p) > \mathrm{sob}_n(l, q).$$

This means that if a point representing $W^{k,p}(\Omega)$ lies above a point representing $W^{l,q}(\Omega)$ and $k > l$, then there exists a continuous linear map from $W^{k,p}(\Omega)$ to $W^{l,q}(\Omega)$, i.e., $W^{k,p}(\Omega)$ is embedded in $W^{l,q}(\Omega)$.

Sobolev spaces corresponding to points on the demarcation line may or may not be embedded in $L^p(\Omega)$. For example, for $n \geq 1$, we have that

$$W^{n,1}(\Omega) \hookrightarrow L^\infty(\Omega) \qquad \text{but } W^{1,n}(\Omega) \not\hookrightarrow L^\infty(\Omega).$$

However, we do have $W^{1,n}(\Omega) \hookrightarrow L^p(\Omega)$ for all $1 \leq p < \infty$. Furthermore, there exists a constant $C(n, \Omega)$, depending on the dimension of Euclidean space and the Lipschitz domain, such that

$$\|v\|_{0,p,\Omega} \leq C(n,\Omega)\, p^{1-1/n} \|v\|_{1,n,\Omega} \quad \text{for all } 1 \leq p < \infty. \tag{8}$$

This inequality (8) provides a precise estimate for comparison of norms.

Example 1.17. By the embedding theorem, we have

(1) $H^1(\Omega) \hookrightarrow C(\bar{\Omega})$ in one dimension (Exercise 1.14)
(2) $H^1(\Omega) \not\hookrightarrow C(\bar{\Omega})$ for $n \geq 2$ (Example 1.13). Namely there is no continuous representation of functions in $H^1(\Omega)$ for $n \geq 2$ and thus the pointwise value is not well defined.
(3) In 2-D, $H^1(\Omega) \hookrightarrow L^p(\Omega)$ for any $1 \leq p < \infty$; in 3-D, $H^1(\Omega) \hookrightarrow L^p(\Omega)$ for $1 \leq p \leq 6$.

For a piecewise smooth function which is also in $H^1(\Omega)$, then it is globally continuous.

Exercise 1.18. *Let Ω, Ω_1 and Ω_2 are three open and bounded domains in $\mathbb{R}^n$. Suppose $\overline{\Omega} = \overline{\Omega}_1 \cup \overline{\Omega}_2$ and $\Omega_1 \cap \Omega_2 = \varnothing$. Let u be a function defined in Ω such that $u|_{\Omega_1} \in C^1(\Omega_1) \cap H^1(\Omega_1)$ and $u|_{\Omega_2} \in C^1(\Omega_2) \cap H^1(\Omega_2)$. Prove that $u \in H^1(\Omega)$ if and only if $u \in C(\Omega)$.*

1.5. **Interpolation theory.** Let H_0 and H_1 be Hilbert spaces with H_1 continuously embedded and dense in H_0. An *intermediate space* H is a space satisfying $H_1 \subset H \subset H_0$. The Hilbert scale provides a way to define a family of intermediate spaces between H_1 and H_0.

Assuming that $(\cdot, \cdot)$ is the inner product on H_0, the space H_1 can be characterized as the domain of an unbounded positive self-adjoint operator $\Lambda : H_1 \to H_0$ that connects the norms by

$$\|u\|_{H_1} = \|\Lambda u\|_{H_0}.$$

For $s \in (0, 1)$, we define the intermediate spaces H_s to be the domain of the operator Λ^s with a norm given by

$$\|u\|_{H_s} = \|\Lambda^s u\|_{H_0}.$$

These spaces provide a scale of function spaces that interpolate between H_1 and H_0 and can be used to study the regularity of solutions to various PDEs.

Example 1.19. Given $-\infty < s_0 < s_1 < \infty$, let $H_0 = H^{s_0}(\mathbb{R}^n)$ and $H_1 = H^{s_1}(\mathbb{R}^n)$. The operator that connects H_0 and H_1 is as follows

$$\Lambda = \mathcal{F}^{-1}(1 + |\xi|^2)^{(s_1 - s_0)/2} \mathcal{F}.$$

In fact, by definition

$$\|v\|_{s_1, \mathbb{R}^n} = \|\Lambda v\|_{s_0, \mathbb{R}^n}.$$

By Hilbert scale, we have

$$[H^{s_1}(\mathbb{R}^n), H^{s_0}(\mathbb{R}^n)]_\theta = D(\Lambda^\theta) = H^{s_\theta}(\mathbb{R}^n) \text{ with } s_\theta = (1-\theta)s_0 + \theta s_1.$$

Let us consider the special case where the spectrum of Λ is discrete and the eigenvectors $\{\varphi_i\}$ form a complete orthonormal basis for H_0. In this case, any $u \in H_0$ can be expanded as

$$u = \sum_{i=1}^{\infty} (u, \varphi_i)\varphi_i.$$

If $u \in H_1$, then we have

$$\Lambda u = \sum_{i=1}^{\infty} \lambda_i (u, \varphi_i)\varphi_i \quad \text{and} \quad \|u\|_{H_1}^2 = \sum_{i=1}^{\infty} \lambda_i^2 (u, \varphi_i)^2.$$

Using these expressions, we define the intermediate space H_s to be the set of elements in H_0 for which the norm

$$\|u\|_{H_s} = \left(\sum_{i=1}^{\infty} \lambda_i^{2s} (u, \varphi_i)^2 \right)^{1/2}$$

is finite.

The definition of H_θ given earlier involves an embedding operator Λ. It is important to note that the definition of H_θ is independent of the choice of Λ. This can be shown using either the K-method or the J-method, as described in the book by Bergh and Löfström [3].

Example 1.20. Assume Ω is a bounded Lipschitz domain. Let $H_1 = H_0^1(\Omega)$ and $H_0 = L^2(\Omega)$. Then H_1 can be viewed as the domain of $\Lambda = (-\Delta)^{1/2}$ with a norm given by

$$\|\nabla v\| = \|\Lambda v\| \quad \text{for all for } v \in H_0^1(\Omega).$$

The interpolated spaces $[H_0^1(\Omega), L^2(\Omega)]_s = H_0^s(\Omega)$ for $s \in (1/2, 1]$.

A simple application of Hölder inequality gives that

Theorem 1.21. *For $u \in H_1$*

$$\|u\|_{H_s} \leq \|u\|_{H_0}^{1-s} \|u\|_{H_1}^{s}$$

Exercise 1.22. *Use integrate by parts to prove the interpolation inequality*

$$\int_\Omega |Du|^2 \, \mathrm{d}x \leq C \left(\int_\Omega u^2 \, \mathrm{d}x \right)^{1/2} \left(\int_\Omega |D^2 u|^2 \, \mathrm{d}x \right)^{1/2}$$

for $u \in H^2(\Omega) \cap H_0^1(\Omega)$.

We now turn to the interpolation of operators between interpolation spaces. Let $\widetilde{H}_1 \hookrightarrow \widetilde{H}_0$ and $H_1 \hookrightarrow H_0$, and further let L be a linear operator such that

$$L : H_0 \mapsto \widetilde{H}_0, \quad \text{with } \|Lu\|_{\widetilde{H}_0} \leq C_0 \|u\|_{H_0} \quad \text{for all } u \in H_0,$$

and

$$L : H_1 \mapsto \widetilde{H}_1, \quad \text{with } \|Lu\|_{\widetilde{H}_1} \leq C_1 \|u\|_{H_1} \quad \text{for all } u \in H_1.$$

A natural question is whether L is bounded from H_θ to $\widetilde{H}_\theta$ for all $\theta \in (0, 1)$.

Theorem 1.23. *Suppose that $H_i, \widetilde{H}_i, i = 0, 1$ and L are given above. Then $L : H_\theta \mapsto \widetilde{H}_\theta$ is also bounded and*

$$\|Lu\|_{\widetilde{H}_\theta} \leq C_0^{1-\theta} C_1^{\theta} \|u\|_{H_\theta} \quad \textit{for all } u \in H_1.$$

1.6. **Trace Theorems.** The trace theorem is used to define function values on the boundary. If $u \in C(\overline{\Omega})$, then $u(x)$ is well-defined for $x \in \partial\Omega$. However, for $u \in W^{k,p}(\Omega)$, the function is defined as an equivalence class of Lebesgue integrable functions, i.e., $u \sim v$ if and only if $u = v$ almost everywhere. Since the boundary $\partial\Omega$ is a set of measure zero (in nth dimensional Lebesgue measure), the pointwise value of $u|_{\partial\Omega}$ is not well-defined for functions in Sobolev spaces.

Theorem 1.24. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth or Lipschitz boundary $\Gamma = \partial\Omega$. Then the trace operator $\gamma : C^1(\overline{\Omega}) \mapsto C(\Gamma)$ can be continuously extended to $\gamma : H^1(\Omega) \mapsto H^{1/2}(\Gamma)$, i.e., the trace inequality holds:*

$$\|\gamma(u)\|_{1/2,\Gamma} \lesssim \|u\|_{1,\Omega}, \quad \textit{for all } u \in H^1(\Omega). \tag{9}$$

Furthermore the trace operator is surjective and has a continuous right inverse.

More general, if Ω is a Lipschitz domain, then the trace operator $\gamma : H^s(\Omega) \to H^{s-1/2}(\Gamma)$ is bounded for $1/2 < s < 3/2$. See McLean [10] (pages 100–106).

1.7. **Norm Equivalence Theorems.** The definition of the norm $\|\cdot\|_{k+1,p}$ involves all the ith derivatives for $i \le k+1$, and among them, the highest derivative is the critical one. Specifically, we can prove that

$$\|u\|_{k+1,p} \eqsim \|u\|_{0,p} + |u|_{k+1,p} \quad \text{for all } u \in W^{k+1,p}(\Omega).$$

Considering this fact for the decomposition

$$W^{k+1,p}(\Omega) = \mathcal{P}_k(\Omega) \oplus (W^{k+1,p}(\Omega)/\mathcal{P}_k(\Omega)),$$

where $\mathcal{P}_k(\Omega)$ denotes the space of polynomials with degree at most k. On the quotient space $W^{k+1,p}(\Omega)/\mathcal{P}_k(\Omega)$, the seminorm $|u|_{k+1,p}$ is indeed a norm.

Lemma 1.25. *For any $u \in W^{k+1,p}(\Omega)$, one has*

$$\inf_{v \in \mathcal{P}_k(\Omega)} \|u + v\|_{k+1,p,\Omega} \eqsim |u|_{k+1,p,\Omega}.$$

In the polynomial space $\mathcal{P}_k(\Omega)$, the norm $\|\cdot\|_{0,p}$ can be replaced by any other norm defined on $\mathcal{P}_k(\Omega)$, since this space is finite-dimensional. More generally, we can consider a semi-norm $F(\cdot)$ on $W^{k+1,p}(\Omega)$ satisfying

$$F(u+v) \le F(u) + F(v), \quad \text{and} \quad F(\alpha\, u) \le |\alpha| F(u).$$

The difference between a norm and a semi-norm is the null space $\ker(F)$. If this space is trivial, i.e., containing only the zero function, then F is a norm. For a semi-norm F on $W^{k+1,p}(\Omega)$ that satisfies $\ker(F) \cap \mathcal{P}_k(\Omega) = \{0\}$, it will define a norm on the subspace $\mathcal{P}_k(\Omega)$.

Theorem 1.26 (Sobolev norm equivalence)**.** *If F is a seminorm on $W^{k+1,p}(\Omega)$ such that $\ker(F) \cap \mathcal{P}_k(\Omega) = \{0\}$, then*

$$\|v\|_{k+1,p} \eqsim F(v) + |v|_{k+1,p} \quad \textit{for all } v \in W^{k+1,p}(\Omega). \tag{10}$$

As a special case of the Sobolev norm equivalence theorem, we present variants of the Poincaré and Friedriches inequalities. Let Γ be a measurable subset of $\partial\Omega$ with positive $n-1$ dimensional Lebesgue measure. By choosing $F(v) = |\int_\gamma v\, ds|$ in (10)), we obtain the Friedrichs inequality:

$$\|v\|_{1,p,\Omega} \lesssim |v|_{1,p,\Omega} + \Big|\int_\Gamma v\, ds\Big| \quad \text{for all } v \in W^{1,p}(\Omega).$$

This inequality implies the Poincaré inequality:

$$\|v\|_{0,p} \lesssim |v|_{1,p,\Omega} \quad \text{for all } v \in W_0^{1,p}(\Omega).$$

Choosing $F(v) = \int_\Omega v\,dx$, we obtain the Poincaré-Friedrichs inequality:

$$\|v\|_{1,p,\Omega} \lesssim |v|_{1,p,\Omega} + \Big|\int_\Omega v\,dx\Big| \quad \text{for all } v \in W^{1,p}(\Omega).$$

As a consequence, if $\bar{v} = \int_\Omega v dx/|\Omega|$ denotes the average of v over Ω, we obtain the average-type Poincaré inequality:

$$\|v - \bar{v}\|_{0,p,\Omega} \lesssim |v|_{1,p,\Omega} \quad \text{for all } v \in W^{1,p}(\Omega).$$

Note that all constants hidden in the notation $\lesssim$ depend on the size and shape of the domain Ω, which is important when applying these inequalities to a simplex with diameter h, as in finite element methods.

2. Elliptic boundary value problems and regularity

In this section, we present the weak formulation of the Poisson equation, as well as its H^2-regularity result on convex or smooth domains.

2.1. Weak formulation of the Poisson equation.

Consider the Poisson equation with homogeneous Dirichlet boundary condition given by

$$-\Delta u = f \text{ in } \Omega, \quad \text{and } u|_{\partial\Omega} = 0, \tag{11}$$

where Ω is a bounded domain in $\mathbb{R}^n$ with smooth boundary. A function $u \in C^2(\Omega) \cap C_0(\Omega)$ that satisfies the Poisson equation (11) is called a *classical solution*. However, the smoothness condition of a classical solution may exclude many interesting solutions for physical problems. To overcome this, we seek a weak solution in a broader space – the Sobolev space. Here, "weak" means that the smoothness is imposed by weak derivatives, which is weaker than the classical notion of derivatives.

The basic idea of Sobolev spaces is to treat functions as functionals. Let us interpret equation (11) in the distribution sense. We seek a solution $u \in \mathcal{D}'(\Omega)$ such that for any $\phi \in C_0^\infty(\Omega)$,

$$\langle -\nabla\cdot\nabla u, \phi\rangle := \langle \nabla u, \nabla\phi\rangle = \langle f, \phi\rangle.$$

This suggests a weak formulation: find $u \in H_0^1(\Omega)$ such that

$$\int_\Omega \nabla u\cdot\nabla\phi = \int_\Omega f\phi, \quad \forall \phi \in C_0^\infty(\Omega). \tag{12}$$

However, we cannot discretize (12) directly since it is difficult to construct a finite-dimensional subspace of $C_0^\infty(\Omega)$.

To extend the action of ∇u on $C_0^\infty(\Omega)$ to a broader space, we define a bilinear form on $H^1(\Omega)\times C_0^\infty(\Omega)$ as follows:

$$a(u,\phi) = \int_\Omega \nabla u\cdot\nabla\phi\,dx.$$

Using the Cauchy-Schwarz inequality, we have

$$a(u,\phi) \le \|\nabla u\|\|\nabla\phi\|.$$

Thus $a(u,\cdot)$ is continuous in the H^1 topology. Since $C_0^\infty(\Omega)$ is dense in $H_0^1(\Omega)$, the bilinear form $a(\cdot,\cdot)$ can be continuously extended to $U\times V := H^1(\Omega)\times H_0^1(\Omega)$. Here, U is called the *trial* space, which is the space we seek a solution in, and V is still the test space.

In (12) the right-hand side $f \in L^2(\Omega) \cong L^2(\Omega)'$. After $C_0^\infty(\Omega)$ is extend to $H_0^1(\Omega)$, we can take $f \in H^{-1}(\Omega) := (H_0^1(\Omega))' = V'$.

Boundary conditions can be imposed by choosing different trial and test spaces. For example:

- homogenous Dirichlet boundary condition:

$$U = H_0^1(\Omega),\ V = H_0^1(\Omega) \tag{13}$$

- homogenous Neumann boundary condition:

$$U = H^1(\Omega)/\mathbb{R},\ V = H^1(\Omega). \tag{14}$$

We are in the position to present an abstract version of the variational (or so-called weak) formulation of the Poisson equation: given an $f \in V'$, find a solution $u \in U$ such that

$$a(u, v) = \langle f, v\rangle \quad \forall\, v \in V. \tag{15}$$

Exercise 2.1. Prove the existence and uniqueness of the weak solution for the Poisson equation with homogenous Dirichlet or Neumann boundary conditions.

2.2. **Regularity theory.** The regularity theory for elliptic equations aims to characterize the smoothness of weak solutions and connect them to classical solutions. This theory serves as a bridge between these two types of solutions and is crucial for the theory of finite element approximation and the convergence of multigrid methods. However, it can be challenging and involves deep analysis. In this section, we provide a brief overview of this theory.

One way to approach regularity theory is to study the space $H(\Delta;\Omega) = \{u \in H^1(\Omega) : \Delta u \in L^2(\Omega)\}$. It is clear that $H^2(\Omega)\cap H_0^1(\Omega) \subset H(\Delta;\Omega)$. However, the reverse inclusion may not hold in general.

We first give a formal derivation for the model problem $-\Delta u = f$ in $\mathbb{R}^n$ and assume u is smooth and vanishes sufficiently rapidly as $|x| \to \infty$ to justify the following integration by parts:

$$\int_{\mathbb{R}^n} f^2 dx = \int_{\mathbb{R}^n} (\Delta u)^2 \,\mathrm{d}x = \sum_{i,j=1}^n \int_{\mathbb{R}^n} \partial_{ii} u \partial_{jj} u \,\mathrm{d}x$$

$$= -\sum_{i,j=1}^n \int_{\mathbb{R}^n} \partial_{iij} u \partial_j u \,\mathrm{d}x = \sum_{i,j=1}^n \int_{\mathbb{R}^n} \partial_{ij} u \partial_{ij} u \,\mathrm{d}x = \int_{\mathbb{R}^n} |D^2 u|^2 \,\mathrm{d}x.$$

We can also see it from Fourier transform for the Laplacian operator on the whole space. Given a distribution v defined on $\mathbb{R}^n$ such that $\Delta v \in L^2$, by the properties of Fourier transform, we have

$$\widehat{D^\alpha v}(\xi) = (i\xi)^\alpha \hat{v}(\xi) = -(i\xi)^\alpha |\xi|^{-2} \widehat{\Delta v}(\xi).$$

The function $(i\xi)^\alpha |\xi|^{-2}$ is bounded by 1 if $|\alpha| = 2$, hence

$$\|\widehat{D^\alpha v}\|_{0,\mathbb{R}^n} \le \|\widehat{\Delta v}\|_{0,\mathbb{R}^n}.$$

By Parseval identity, we have

$$\|D^\alpha v\|_{0,\mathbb{R}^n} \le \|\Delta v\|_{0,\mathbb{R}^n} \quad \text{for all } |\alpha| = 2.$$

The above inequality illustrates an important fact that if v is a function such that $\Delta v \in L^2$, then all its second order derivatives are also in L^2. That is the Frobenius norm of the Hessian matrix can be bounded by its trace which is true if looking at the Fourier transform.

This is a rather significant fact since Δv is a very special combination of the second order derivatives of v.

Smooth and bounded domains. The proofs described above can be extended to bounded domains with smooth boundaries, although this extension is not trivial. The following theorem is well-known and can be found in most textbooks on elliptic boundary value problems:

Theorem 2.2. *Let Ω be a smooth and bounded domain of $\mathbb{R}^n$. Then for each $f \in L^2(\Omega$, there exists a unique $u \in H^2(\Omega) \cap H_0^1(\Omega)$ satisfying $-\Delta u = f$ and*

$$\|u\|_{2,\Omega} \leq C\|f\|_{0,\Omega}$$

where C is a positive constant depending on Ω.

For general domains, if $f \in L^2(\Omega)$, then the weak solution u satisfies $u \in H^2_{\mathrm{loc}}(\Omega)$. This means that for every domain $U \subset\subset \Omega$, we have $u \in H^2(U)$. To prove this local result, we use a difference quotient to define a discrete approximation of D^2u. The condition $U \subset\subset \Omega$ allows us to choose a smooth cut-off function away from the boundary. Along the boundary, we use a suitable change of coordinates to prove that all tangential derivatives exist similarly. For the normal derivative, we use the equation to bound it by f and a combination of the other derivatives. Since the boundary of Ω is smooth enough, we can find a partition of unity and glue the local charts together. A detailed proof can be found, for example, in [7].

Lipschitz domains. It is important to note that Theorem 2.2 does not hold for general Lipschitz domains. The smoothness of $\partial\Omega$ is necessary to establish the regularity of the solution. In fact, on Lipschitz domains, the regularity of u may depend on the shape of Ω. For instance, the gluing of boundary pieces at corner points can be a challenging technical issue that may affect the regularity of the solution. In the following example, we illustrate how the shape of Ω can influence the regularity of u.

Example 2.3. Given $\beta \in (0,1)$, consider the following non-convex domain

$$\Omega = \{(r,\theta) : 0 < r < 1, 0 < \theta < \pi/\beta\}.$$

Let $v = r^\beta \sin(\beta\theta)$. Being the imaginary part of the complex analytic functin z^β, v is harmonic in Ω. Define $u = (1-r^2)v$. A direct calculation shows that

$$-\Delta u = 4(1+\beta)v \ \text{ in } \ \Omega, \quad \text{and } \ u|_{\partial\Omega} = 0.$$

Note that $4(1+\beta)v \in L^\infty(\Omega) \subset L^2(\Omega)$, but $u \notin H^2(\Omega)$.

Nevertheless, a slightly weaker result does hold for general Lipschitz domains.

Theorem 2.4. *Assume that Ω is a bounded Lipschitz domain. Then there exists a constant $\alpha \in (0,1]$ such that*

$$\|u\|_{1+\alpha} \leq C\|f\|_{\alpha-1}, \tag{16}$$

for the solution u of (15), where C is a constant depending on the domain Ω.

Again a proof for the above regularity result is quite complicated, we refer to [9, 5, 2].

Convex domains. A remarkable fact is that we can take $\alpha = 1$ in the above theorem for convex domains. This means that Theorem 2.2 can be extended to convex domains.

Theorem 2.5. *Let Ω be a convex and bounded domain of $\mathbb{R}^n$. Then for each $f \in L^2(\Omega)$, there exists a unique $u \in H^2(\Omega)$ to the solution of (15)) that satisfies*

$$\|u\|_{2,\Omega} \le C\|f\|_{0,\Omega}$$

where C is a positive constant depending only on the diameter of Ω.

We outline a proof for a convex polygon and refer to [13] for details. First, we approximate the convex polygon from the interior by a sequence of smooth and convex domains Ω_k. On each Ω_k, we establish a H^2-regularity result $|u_k|_2 \le C(\Omega_k)\|f\|$ for the solution $-\Delta u_k = f$ in Ω_k and $u_k = 0$ on $\partial\Omega_k$. Then we take a limit in an appropriate topology to obtain the desired result on Ω. To be able to take the limit, the constant $C(\Omega_k)$ should be uniformly bounded. This requires a careful study of the regularity result on Ω_k.

One useful identity for this study is the refined integration by parts, cf. [8],

$$\int_U (\Delta u)^2 \,\mathrm{d}x = \sum_{i,j=1}^{n} \int_U |\partial_{ij} u|^2 \,\mathrm{d}x + \int_{\partial U} \operatorname{div} \boldsymbol{\nu} \left|\frac{\partial u}{\partial \boldsymbol{\nu}}\right|^2 \,\mathrm{d}S, \tag{17}$$

where $\boldsymbol{\nu}$ is the outwards unit normal vector of ∂U. When U is C^2 and convex, $\operatorname{div}\boldsymbol{\nu} = \operatorname{trace}(D\boldsymbol{\nu}) \ge 0$, and thus the last term is positive, leading to $|D^2 u|_U \le |\Delta u|_U$. Lower order terms in the H^2-norm can be bounded by Poincar'e inequality, and the constant depends only on the diameter $\operatorname{diam}(\Omega_k) \le \operatorname{diam}(\Omega)$ uniformly.

Exercise 2.6. Prove identity (17) using the identity

$$-\Delta \boldsymbol{f} = -\operatorname{grad}\, \operatorname{div} \boldsymbol{f} + \operatorname{curl} \operatorname{curl} \boldsymbol{f},$$

and then integration by parts by multiplying $\boldsymbol{f} = \nabla u$ to both sides and integrate over U.

Concave polygons. In the case of concave polygons, full regularity may not hold. There are two approaches to obtain weaker regularity results for such domains.

The first approach, as shown in [9, 5, 2], is to use fractional Sobolev spaces. Specifically, we consider $-\Delta : H_0^1(\Omega) \cap H^{1+s}(\Omega) \to H^{-1+s}(\Omega)$, which holds for any $s \in [0, s_0)$ for some $s_0 \in (0, 1]$ that depends on Ω.

The second approach is to use L^p spaces instead of L^2. In particular, it can be shown that (cf. [8])

$$-\Delta : H_0^1(\Omega) \cap W^{2,p}(\Omega) \to L^p(\Omega) \tag{18}$$

is an isomorphism for any $p \in (1, p_0)$, where $p_0 > 1$ depends on the domain Ω. We will discuss this further in the context of adaptive finite element methods.

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