

Real Analysis Qualifying Exam

September 20, 2024

Student's math exam ID#: _____

INSTRUCTIONS: Do all work on the sheets provided. There is a blank page following each problem. **Please do not use the back of the sheets in your solutions.**

Problem	Point Value	Points Received
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
Total	60	

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Problem 1 (10 points)

Let f be a positive continuous function on $[0, 1]$. Prove that

$$\lim_{r \rightarrow -\infty} \left(\int_0^1 f^r \right)^{1/r} = \min_{[0,1]} f.$$

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Problem 2 (10 points)

Suppose that $f \in L^1(\mathbb{R})$. Show that

$$\lim_{n \rightarrow \infty} f(x + n) = 0 \text{ for a.e. } x \in \mathbb{R}.$$

Here $n = 1, 2, 3, \dots$

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Problem 3 (10 points)

Fix $p > 1$, and suppose that $u \in L^p([0, 1])$ is nonnegative and that

$$\int_{\{u \geq t\}} u \, dx \leq \min\{1, t^{-p}\}$$

for all $t > 0$. Prove that

$$\int_0^1 u^p \, dx \leq p.$$

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Problem 4 (10 points)

Let

$$g(x) = \begin{cases} 1, & x \in [k, k + 2/3], k \in \mathbb{Z} \\ -1, & \text{otherwise.} \end{cases}$$

Let $g_j(x) = g(jx)$, $j \in \mathbb{N}$. Prove that for all $f \in L^1([0, 1])$,

$$\lim_{j \rightarrow \infty} \int_0^1 g_j(x) f(x) dx = \frac{1}{3} \int_0^1 f(x) dx.$$

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Problem 5 (10 points)

Show that there does NOT exist a function $g \in L^1(\mathbb{R})$ such that

$$f * g = f$$

for all $f \in L^1(\mathbb{R})$. Here $f * g(x) = \int_{\mathbb{R}} f(y)g(x - y) dy$ is the convolution of f and g .

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Problem 6 (10 points)

Let $f \in L^1(\mathbb{R})$, and for all $z \in \mathbb{R}$ define

$$g(z) := \sup_{r>0} \frac{1}{2r} \int_{z-r}^{z+r} |f| \, dx.$$

Prove that, for any $t > 0$,

$$|\{z : g(z) > t\}| \leq \frac{5}{t} \|f\|_{L^1(\mathbb{R})}.$$

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