

COMPREHENSIVE EXAM IN REAL ANALYSIS

Tuesday, June 24, 2024 — 1:00-3:30pm

Problem	1	2	3	4	5	6	7	8	9	Σ
Points										

Each problem is worth 10 points. No books, notes, or calculators are allowed.

Student's code:

Problem 1.

Let $\{a_n\}_{n=0}^{\infty}$ be a sequence of positive numbers and $\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = a$. Show that the limit $\lim_{n \rightarrow \infty} a_n^{1/n}$ exists and has the same value a .

Problem 2.

For each $q > 0$, investigate convergence/divergence of the integral

$$\int_1^{\infty} \frac{x}{1 + x^q |\sin(x)|} dx$$

Problem 3.

Let A be a non-empty subset of $\ell^\infty(\mathbb{N})$. Define $f : \ell^\infty(\mathbb{N}) \rightarrow \mathbb{R}$ by

$$f(x) \stackrel{\text{def}}{=} \inf_{y \in A} \|x - y\|_{\ell^\infty}.$$

Prove that f is Lipschitz with constant 1, i.e. that for any $x, y \in \ell^\infty(\mathbb{N})$ we must have

$$|f(x) - f(y)| \leq \|x - y\|_{\ell^\infty}$$

Problem 4.

Let $E \subset \mathbb{R}$ be a bounded set, and $f : E \rightarrow \mathbb{R}$ be a uniformly continuous function. Prove that f is bounded on E .

Problem 5.

Let X be a compact metric space and let $f : X \rightarrow \mathbb{R}$ and $f_n : X \rightarrow \mathbb{R}$ be continuous functions for $n \in \mathbb{N}$. Assume that $f_n \leq f_{n+1}$ for all $n \in \mathbb{N}$ and that f_n converges pointwise to f . Show that the convergence is uniform and that this claim is not true in general if f is not assumed to be continuous.

Problem 6.

Let $U \subset \mathbb{R}^2$ be an open set $U = \{(x, y) \mid y > \arctan x > 0, x > \arctan y > 0\}$.
For which $a > 0$ the integral $\int_U f_a(x, y)$ converges, if $f_a(x, y) = \frac{1}{(x+y)^a}$?

Problem 7.

Find the flux of the vector field $\begin{bmatrix} x^2 \\ y^2 \\ z^2 \end{bmatrix}$ in $\mathbb{R}^3$ through the surface of the unit sphere, oriented by the outward-pointing normal.

Problem 8.

Show that the mapping $F(x, y) = (e^x + e^y, e^x + e^{-y})$ is locally invertible at every point $(x, y) \in \mathbb{R}^2$.

Problem 9.

Show that the set $\mathbb{N}$ of natural numbers can be represented as a union of uncountably many distinct subsets $\mathbb{N} = \bigcup_{\alpha \in A} N_\alpha$ in such a way that for any $\alpha, \beta \in A$ either $N_\alpha = N_\beta$, or the intersection $N_\alpha \cap N_\beta$ is finite.