

Complex Analysis Qualifying Exam

September 17, 2024

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Problem 1: Find all real-valued harmonic functions u on the whole complex plane such that for all $z = x + iy \in \mathbb{C}$, one has:

$$u(z) \leq x^3 - 3xy^2 + x + y.$$

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Problem 2: Let

$$f(z) = \sum_{n=1}^{\infty} \frac{n^n}{(n!)^2} z^n.$$

- (a) Prove that f is an entire function;
- (b) Evaluate the following integral:

$$\int_{|z|=1} f\left(\sin\left(\frac{1}{z}\right)\right) dz.$$

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Problem 3: Let P be a polynomial of degree n , and let

$$M(r, P) = \sup_{|z|=r} \{|P(z)|\}, \quad r > 0.$$

Prove that the function $r \mapsto M(r, P)$ is increasing and the function $r \mapsto \frac{M(r, P)}{r^n}$ is decreasing.

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Problem 4: Prove that the function

$$f(z) = \int_0^1 \frac{\cos(t)}{t-z} dt + \int_2^3 \frac{\sin(t)}{t-z} dt$$

admits a Laurent expansion in the annulus $\{1 < |z| < 2\}$ and compute this expansion.

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Problem 5: Let a and b be two complex numbers and assume that b is not equal to any integer ≤ 0 . Show that the radius of convergence of the power series

$$\sum_{n=1}^{\infty} \frac{a(a+1)\cdots(a+n)}{b(b+1)\cdots(b+n)} z^n$$

is at least 1 and can be infinity in some cases.

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Problem 6: Let $\alpha < 1$ and $\mathcal{F}$ be the family of all power series $\sum_{n=1}^{\infty} a_n z^n$ with coefficients in $\mathbb{C}$ such that for all n :

$$|a_n| \leq e^{n^\alpha}.$$

Show that such power series converge on the unit disk $D(0, 1)$ and they form a normal family on the unit disk. Is the statement still true if $\alpha = 1$?

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Problem 7: Let f be holomorphic on the disk $D(0, R)$, for some $R > 1$. Show that there exists $z_0 \in \mathbb{C}$ with $|z_0| = 1$ such that

$$\left| f(z_0) - \frac{1}{z_0} \right| \geq 1.$$

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Problem 8: Let f be an entire function with $f(0) = 1$, which has at least n zeros in the unit disk $D(0, 1)$. Show that for each $R > 0$, there exists a point z_R on the boundary of the disk $D(0, R)$ such that

$$|f(z_R)| \geq R^n.$$

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Problem 9: Let $\{f_n\}$ be a bounded sequence of holomorphic functions on the unit disk $D = D(0, 1)$ and let $\{z_m\}$ be a sequence in D that converges in D . Assume that $\lim_{n \rightarrow \infty} f_n(z_m)$ exists for all m . Show that $\{f_n\}$ converges uniformly on compact subsets of D .