

Complex Analysis Qualifying Exam

June 20, 2024

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Problem 1: Let u be a real-valued harmonic function on a domain $U \subset \mathbb{C}$. Show that the zero set of u does not have an isolated point. Recall that the zero set is defined as $Z = \{z \in U; u(z) = 0\}$.

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Problem 2: Is there an entire function f that satisfies $|f(z)| \geq e^{|z|}$ for all values of $|z|$ large enough, that is for all z with $|z| \geq M$ for some large positive M ? Either provide an example or prove that none exists.

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Problem 3: Fix a complex number λ with $0 < |\lambda| < 1$. For $n \geq 1$, let

$$f(z) = (z - 1)^n e^z + \lambda(z + 1)^n.$$

Show that f has exactly n simple zeros in the right-half plane $\operatorname{Re}(z) > 0$.

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Problem 4: Show that $\cosh\left(z + \frac{1}{z}\right)$ accepts the following Laurent series centered at 0:

$$\cosh\left(z + \frac{1}{z}\right) = a_0 + \sum_{n=1}^{\infty} a_n \left(z^n + \frac{1}{z^n}\right), \quad z \neq 0,$$

where

$$a_n = \frac{1}{2\pi} \int_0^{2\pi} \cos(nt) \cosh(2 \cos(t)) \, dt, \quad n = 0, 1, 2, \dots$$

Recall that $\cosh(z) = \frac{1}{2}(e^z + e^{-z})$.

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Problem 5: Let $f(z)$ be a holomorphic function on the punctured disk $D(0, 1) \setminus \{0\}$ and g be a non-constant entire function. Determine the type of the singularity of $g \circ f$ at $z = 0$ for each of the following cases (with proofs):

- (a) $z = 0$ is removable singularity of f ;
- (b) $z = 0$ is a pole of f ;
- (c) $z = 0$ is an essential singularity of f .

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Problem 6: Let $\mathcal{F}$ be the family of holomorphic functions f on the unit disk $D(0, 1)$ satisfying:

$$|f(0)|^2 + \iint_{D(0,1)} |f'(z)|^2 (1 - |z|^2) dA \leq 1.$$

Prove that $\mathcal{F}$ is a normal family on $D(0, 1)$.

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Problem 7: Let g be a non-constant entire function. Prove or disprove that there is a holomorphic function $f(z)$ on the unit disk $D(0,1)$ such that

$$f(z) = g(\bar{z}), \quad \text{if } |z| = \frac{1}{2}.$$

What if we only assume that g is a non-constant holomorphic function on the punctured plane $\mathbb{C} \setminus \{0\}$?

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Problem 8: The Sterling formula reads $n! = n^n e^{-n} s_n$, where $\lim_{n \rightarrow \infty} s_n^{1/n} = 1$. Use the Sterling formula to find the radius of convergence of the following power series:

$$\sum_{n=1}^{\infty} \frac{(n!)^3}{(3n)!} z^n.$$

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Problem 9: Let $P(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_0$. Prove that either $P(z) \equiv z^n$ or there exists a point ζ on the unit circle such that $|P(\zeta)| > 1$.

Hint: Note that $Q(z) := z^n P\left(\frac{1}{z}\right)$ is a polynomial.